Vorticity Redistribution and Vortex Ring Development Analysis in Boundary Layer Transition

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Both laminar and turbulent flows have vorticity. Vorticity of laminar flow is irrotational and concentrated near the wall surface, but vorticity of turbulent flow is mainly rotational with different sizes of vortices and distributed widely inside the boundary layer. Flow transition is really a process of vorticity redistribution in a boundary layer. In this paper, we mainly study how vorticity rolls up from the solid wall surface and how irrotational vorticity (shear layer) becomes rotational (vortex). The nature of turbulence generation is that fluid cannot tolerate high shear and shear must transfer to rotation. In this paper, a detailed analysis is given about how shear transfers to fast rotation cores. The paper also shows the rotational vortex would have minimum energy dissipation, and therefore, it is a stable state which means turbulence dominated by rotation, is a stable state.

Keywords: Vorticity, Vortex, Redistribution, Rotation, Flow Transition

Nomenclature

\[ M_\infty = \text{Mach number} \]
\[ \delta_m = \text{inflow displacement thickness} \]
\[ T_\infty = \text{free stream temperature} \]
\[ L_{z_{in}} = \text{height at inflow boundary} \]
\[ L_x = \text{length of computational domain along x direction} \]
\[ L_y = \text{length of computational domain along y direction} \]
\[ x_m = \text{distance between leading edge of flat plate and upstream boundary of computational domain} \]
\[ A_{2d} = \text{amplitude of 2D inlet disturbance} \]
\[ A_{3d} = \text{amplitude of 3D inlet disturbance} \]
\[ \omega = \text{frequency of inlet disturbance} \]
\[ \alpha_{2d}, \alpha_{3d} = \text{two and three dimensional streamwise wave number of inlet disturbance} \]
\[ \beta = \text{spanwise wave number of inlet disturbance} \]
\[ \gamma = \text{ratio of specific heats} \]
\[ \mu = \text{viscosity} \]

I. Introduction

Turbulence is still an unsolved scientific problem, which is not only important to science but also to industrial applications in aerospace engineering, mechanical engineering, energy engineering, bio

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engineering and many others. Yes, turbulence remains the most important unsolved problem of classical physics. Clearly understanding of turbulence will help scientists and engineers cope with the broad range of turbulent flows.

1.1 A short review of study on late boundary layer transition

The transition process from laminar to turbulent flow in boundary layers is a basic scientific problem in modern fluid mechanics. After over a hundred years of study on flow transition, the linear and weakly non-linear stages of flow transition are pretty well understood (Herbert, 1988; Kachanov, 1994). However, for late non-linear transition stages, there are still many questions remaining for research (Kleiser et al., 1991; Sandham et al., 1992; U. Rist et al., 1995; Borodulin et al., 2002; Bake et al., 2002; Kachanov, 2003). Adrian (2007) gave a review about the hairpin vortex organization in boundary layers in which he realized the importance of low speed streaks, ejections (Q2) and sweeps (Q4) but did not discuss the shear layers caused by sweeps and ejections and the role of the shear layer instability. He also believes the multiple vortex rings in a vortex packet is produced by auto-generation and the asymmetric structure is generated in each packet. He says the vortex is attached first and then detached from the wall. Wu and Moin (2009) reported a new DNS for late flow transition on a flat plate. They did obtain fully developed turbulent flow with structure of forest of hair-pin vortices by flow transition at zero pressure gradients. However, they did not give the mechanism of the late flow transition. Actually, similar work for the whole process of K- and H-type transition has been reported by Liu et al. (1995, 1996, 1997) and Rist et al. (2002, JFM). The newer results have higher resolution, but all reported vortex structures are similar.

1.2 A short review of study on turbulence coherent structure

For the turbulence coherent structure, a short review to cite some important work in study of the turbulence coherent structure is given here. Turbulence hairpin vortex was found very early (Theodorsen, 1952). A well-known visualization of a low Reynolds number turbulent boundary layer was given by Falco (1977) which illustrates several of the known types of coherent structures. For instance, the patterns that are most visible reside near the boundaries between clear regions and smoke-filled regions, mainly near the outer edge, where one can see the large-scale motions (LSM) or turbulent bulges. Perry and Chong (1982) made the most serious theoretical usage of Theodorsen’s horseshoe vortex paradigm by constructing a model of individual hairpins scattered randomly in the streamwise-spanwise plane and containing a hierarchy of sizes. In their model, the location of each hairpin was statistically independent of every other hairpin. With appropriately chosen parameters for the hairpin geometry and the distribution of the hierarchy, the model reproduced many aspects of statistical quantities such as mean velocity, Reynolds stresses, and spectra. They imagined the wall layer as a forest of “hairpin” or “horseshoe” vortices to be modeled with simplified-shapes in a hierarchy of scales above the wall, but all attached to it.

The near-wall streaky structure with high- and low-speed regions aligned in the streamwise direction was visualized in the developed turbulent boundary layer by Kline et al. (1977). They suggested that the near-wall low-speed streaks with an average spanwise spacing of about 100 wall units play an important role in generating the turbulent energy through a sequence of bursting events. In developed wall turbulence, on the other hand, on the basis of the results of direct numerical simulations (DNS), Robinson (1991) identified asymmetric hairpin (or horseshoe) vortices and quasi-streamwise vortices as two dominant coherent vortices in the wall turbulence, quasi-streamwise vortices close to the wall and hairpin vortices in the log-law region. The mechanisms of the vorticity rollup, the large vortex formation, and the small length scale generation were also discussed by other authors (Davidson, 2004; Brandt and Henningsson, 2002; Lee and Wu, 2008; Wallace, 2013).

1.3 DNS and high resolution experiment oppose the concept of “vortex breakdown”

According to current flow transition theory, the flow transition process has been described as 1) receptivity, 2) linear instability, 3) non-linear growth and interaction, 4) breakdown to turbulence. However, the authors believe that turbulence is not caused by “vortex breakdown” but “vortex buildup” and linear modes only play a role to trigger vorticity rollup, but not directly cause the flow transition.
Therefore, the authors believe that the transition process should be described as 1) perturbation and growth (which may include linear modes or other disturbances), 2) large vortex formation including vorticity rollup and shear layer instability, 3) multiple level vortex structure buildup including sweeps, ejections and small length scale generation, 4) symmetry loss and being chaotic to turbulence (Liu et al., 2014).

First, contradicting to current transition theory, there is no “vortex breakdown” but there is a “turbulence vortex structure buildup,” which is just the opposite. “Vortex breakdown” is theoretically incorrect and is never observed by any experiment or DNS. Right now, most flow transition papers just use one word, “vortex breakdown”, to describe the last stage of flow transition. If “vortex breakdown”, which never exists, means flow transition from the laminar state to turbulence state, the authors believe that we need more than one hundred research papers to describe such a process not only one word “vortex breakdown.” According to Dr. Cai’s high resolution experiment (Figure 1, Personal contact) with the highest resolution of 1 \( \mu m \), while most of our experiments only have mm-scale resolution, large vortices interaction could produce countless small vortices with the scale in the order of 1 \( \mu m \), but did not find any large vortex breakdown, even no large vortex deformation. The large vortices are still alive, which contradicts to Richardson’s large vortex short turnover time, expected by \( l/u \) (Davidson, 2004). Cai et al. conclude that the classical and current theory, that small length vortices are produced by large vortex breakdown and the energy is passed from large vortex to smaller vortex through “vortex breakdown”, has no way to be correct. The other impressive qualitative agreement is that the vortex rings fast rotate with a rotation speed of around 10,000 circles per second in a jet flow while the our DNS shows the rotation speed is around 8,000 circles per second in a boundary layer.

![Figure 1. Vortices generated by water jet (Cai’s experimental observation with highest resolution of 1 \( \mu m \), personal contact)](image)

Wallace (2013) pointed out in his review paper: “… there has been remarkable progress in turbulent boundary layer research in the past 50 years, particularly in understanding the structural organization of the flow. Consensus exists that vortices drive momentum transport but not about the exact form of the vortices or how they are created and sustained.” The authors have conducted a new high order DNS with large number of grids to study the “turbulence generation and sustenance” and try to give exact form of the vortices or how they are created and sustained.

Since the authors believe turbulence is generated by “vortex buildup” but not “vortex breakdown”, and believe the nature of turbulence generation is that fluid cannot tolerate high shear and shear layer must transfer to fast-rotating vortex cores, the vorticity redistribution, vorticity rollup, shear layer transfer to rotation and the fast-rotating core formation is focused in the current paper.
II. DNS case setup and validation

In order to get deep understanding on the mechanism of the late stages of flow transition in a boundary layer and physics of turbulence, Liu et al. recently conducted a high order direct numerical simulation (DNS) with medium size of computation, i.e. 60 million (1920×241×128) gird points and about 600,000 time steps, to study the mechanism of the late stages of flow transition in a boundary layer at a free stream Mach number 0.5 (Chen et al., 2009, 2010a, 2010b, 2011a, 2011b; Liu et al., 2010a, 2010b, 2010c, 2010d, 2011a, 2011b, 2011c, 2013, 2014; Lu et al., 2011a, 2011b, 2012, 2013; Yan et al., 2013, 2014). In their preliminary DNS calculation, a number of new observations are made including:

1) Mechanism of spanwise vorticity rollup
2) Mechanism of transfer from flow shear to rotation
3) Mechanism of spanwise vortex tube formation and role of the linear unstable modes
4) Mechanism of Λ−vortex root formation
5) Mechanism of low speed zone and first ring-like vortex formation
6) Mechanism of multiple vortex ring formation
7) Mechanism of second sweep formation
8) Mechanism of positive spike formation
9) Mechanism of second level high share layer formation
10) Mechanism of secondary and tertiary vortex formation
11) Mechanism of U-shaped vortex formation
12) Mechanism of small length vortices generation
13) Mechanism of multiple level high shear layer formation
14) Mechanism of energy transfer paths from the large length scale to the small ones
15) Mechanism of symmetry loss and flow chaos
16) Mechanism of thickening of turbulence boundary layer
17) Mechanism of high surface friction of turbulent flow

2.1 Case setup

The computational domain is displayed in Figure 2. The grid level is 1920×128×241, representing the number of grids in streamwise (x), spanwise (y), and wall normal (z) directions. The grid is stretched in the normal direction and uniform in the streamwise and spanwise directions. The length of the first grid interval in the normal direction at the entrance is found to be 0.43 in wall units (Z+=0.43). The flow parameters, including Mach number, Reynolds number, etc. are listed in Table 1. Here, $\frac{x_m}{\delta} = 300.796$, $\frac{L_x}{2.99\delta} = 40\delta$, $\frac{L_y}{2.33\delta} = \frac{L_z}{40\delta}$ are the lengths of the computational domain in x, y, and z-directions, respectively, and $\frac{T_w}{273.15K}$ is the wall temperature. Table 1 provides the data for the case setup.

![Figure 2. Computation domain](image)

Table 1. Flow parameters
<table>
<thead>
<tr>
<th>$M_\infty$</th>
<th>$Re$</th>
<th>$x_m$</th>
<th>$L_x$</th>
<th>$L_y$</th>
<th>$L_{z_m}$</th>
<th>$T_w$</th>
<th>$T_{\infty}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>1000</td>
<td>300.79 $\delta_m$</td>
<td>798.03 $\delta_m$</td>
<td>22 $\delta_m$</td>
<td>40 $\delta_m$</td>
<td>273.15K</td>
<td>273.15K</td>
</tr>
</tbody>
</table>

### 2.2 DNS validation

The DNS code has been validated (Jiang et al., 2003; Liu et al., 2010a, 2011b) carefully by UTA and NASA Langley researchers to make sure the DNS results are correct. The PI’s new DNS results have been compared with the liner theory, experiment and other DNS reports. The agreement is very good.

All these verifications and validations made by NASA and UTA researchers show that our DNS results are reliable.

### III. Some new understandings on fundamental fluid mechanics

Fluid mechanics tells us vorticity cannot be generated inside the flow field for low speed flow. Flow transition is really a process of vorticity redistribution and the vorticity source is the original Blasius solution. The vorticity redistribution is performed by vorticity rollup from the wall and non-rotating vorticity (shear) transfers to rotating vorticity (vortex) through shear layer instability. The external disturbance has very small vorticity magnitude which is negligible and cannot be the source of turbulence vortex. Liu et al. believe burning wood can make bread, but bread is not made by wood but flour. The linear unstable modes and other external perturbations are wood, while the original Blasius solution (laminar boundary layer) is the flour. It is clear that the role of the linear unstable modes is to trigger the vorticity rollup and vorticity redistribution. Flow transition from laminar to turbulent is a process of vorticity redistribution. In turbulence, vorticity distribution is not concentrated near the wall any more, but whole boundary layer and vorticity is not all irrotating vorticity like shear, but mainly rotating vorticity like vortex. Linear modes suppression tries to reduce or eliminate these modes, but turbulence does not consist of these modes but rotating vortices due to vorticity redistribution. Other perturbations could cause vorticity rollup and vorticity redistribution as well. That is the reason why linear mode suppression including the roughness study is hard to be successful. Several decades and tens of millions dollars have been spent for such a technology, but the real question is what the payoff is? According to Liu’s theory, there is no hope for such a technology because turbulence is not caused by these unstable modes directly, but vorticity rollup and fluid inherent property that fluids cannot tolerate strong shear and shear must transfer to fast rotation cores. There is no T-S or other linear modes when the base flow is changed in very early stage of flow transition while the base velocity profile is changed, inflection points are developed and vortex structure is forming.

#### 3.1 Definition of vortex

It is unfortunate that people believe vortex is a vortex tube (e.g. Davidson, 2004), but it should not be. Intuitively, the vortex what people see is not a vortex tube, but a rotation core. It is a serious mistake to consider vortex as a vortex tube. In general, vortex tube does not allow vorticity line to penetrate, but vortex has many vortex lines which can penetrate the vortex surface, get in the vortex and then get out from the vortex. Therefore, vortex cannot be a vortex tube. Since it is a rotational core, Lambda 2 or Q-criteria can be used to identify the vortex. However, as pointed out by Liu et al., these iso-surface based methods may give a fake “breakdown” by choosing inappropriate value for the iso-surface. We then turn to vortex filaments. A “vortex” must be a place where a rotation core is formed.

#### 3.2 Vorticity and rotation

In English, vorticity is almost considered as rotation, but they are two different concepts although rigid body rotation always has vorticity, but vorticity does not mean pure rotation like rigid body. In general, when $\nabla \times \vec{V} = 0$, we call the flow irrotational. If $\nabla \times \vec{V} \neq 0$, we call the flow is rotational. However, this is a serious misunderstanding. When $\nabla \times \vec{V} \neq 0$, the flow could still be irrotational like a laminar
boundary layer on a flat plate (Blasius solution). $\nabla \times \vec{V}$ is very large near the wall, but the flow is irrotational.

### 3.3 Revisit of Helmholtz velocity decomposition

According to Helmholtz, fluid motion can be decomposed into two parts, rotational part like rigid body and shear part like elastic body if we ignore the translation part (Fig. 3):

$$\vec{V}(\vec{X} + d\vec{X}) = \vec{V}(\vec{X}) + d\vec{V}$$

$$d\vec{V} = d\vec{X} \cdot \nabla \vec{V}$$

$$\nabla \vec{V} = \frac{1}{2} (\nabla \vec{V} + \nabla \vec{V}^T) + \frac{1}{2} (\nabla \vec{V} - \nabla \vec{V}^T) = \vec{\varepsilon} + \frac{1}{2} (\nabla \vec{V} - \nabla \vec{V}^T)$$

$$d\vec{V} = \frac{1}{2} (\vec{\varepsilon} - d\vec{X} \times \vec{\omega}), \text{ where } \vec{\omega} = \frac{1}{2} \nabla \times \vec{V}$$

FIGURE 3: Helmholtz velocity decomposition of a fluid Parcel

People in general believe that the first symmetric part is deformation and the second anti-symmetric part (skew part) is rotation. However, Helmholtz's velocity decomposition is a misunderstanding. That is not the case because the second part is vorticity but not rotation. Vorticity should be decomposed again into two parts. One is rotational vorticity like rigid body (pure rotation) which is something like vortex core. The other part is defined by the authors as non-rotational vorticity (vorticity without rotation) something like laminar boundary layer (Blasius solution). Pure rotating flow has vorticity, but vorticity does not mean rotation. For example, if $\nabla \times \vec{V} = \vec{0}$, the flow is irrotational, but if $\nabla \times \vec{V} \neq \vec{0}$ or even $\nabla \times \vec{V}$ is very large, does it mean flow is rotating? No, the flow could still be irrotational like Blasius solution. The anti-symmetric part should be decomposed as two parts: $\frac{1}{2} \nabla \times \vec{V} = \Omega \vec{R} + \frac{1}{2} \nabla \times \vec{V} - \Omega \vec{R} = \Omega \nabla \times \vec{V} + (1 - \Omega) \nabla \times \vec{V}$, which we can call rotational vorticity and non-rotational vorticity respectively. $\Omega$ is a ratio of rotational part over vorticity. The flow transition really means the non-rotational vorticity part transfers to rotational part while the vorticity flux keeps conserved. On the other hand, the deformation part cannot transfer to rotational part directly, but they can be dissipated due to the viscosity. Yes, the total flow can be decomposed to three parts as translation, deformation and vorticity or, equivalently, translation, symmetric tensor and anti-symmetric (skew)
tenser, but we cannot say the fluid velocity can be decomposed to translation, deformation and rotation which is a serious mistake in fundamental fluid dynamics. After revisiting Helmholtz velocity decomposition, we believe the correct answer for the decomposition of a fluid particle motion should be described as follows. We can decompose the fluid motion to four parts: translation, deformation, rotational vorticity, and non-rotational vorticity since the vorticity includes the rotational and non-rotational parts. Vorticity does not mean rotation although rigid body rotation must have vorticity. In general, rotation is not directly proportional to vorticity, which means that in the place where rotation is strong, vorticity could be small and in the place where vorticity is large, rotation could be weak or void. The transition or turbulence generation is a process of that the non-rotational vorticity transfers to rotational vorticity. In English, rotation and vorticity have almost same meaning, which actually is confusing in science.

3.4 Calculation of rotation ratio $\Omega$

There is a question, how to find the rotational part? Let us take 2-D velocity decomposition first and the 3-D velocity decomposition is very similar. In general, a velocity gradient tensor $\nabla \vec{V}$ of a fluid particle can be decomposed as symmetric and anti-symmetric parts, representing deformation and vorticity respectively. However, vorticity is not exactly rotation and should be further decomposed to rotational and irrotational vorticity.

Assume $A$ represents the velocity gradient tensor:

$$A = \nabla \vec{V} = \frac{1}{2} (\nabla \vec{V} + \nabla \vec{V}^T) + \frac{1}{2} (\nabla \vec{V} - \nabla \vec{V}^T)$$

in a $x-z$ plane:

$$\begin{bmatrix}
\frac{\partial u}{\partial x} & \frac{\partial u}{\partial z} \\
\frac{\partial w}{\partial x} & \frac{\partial w}{\partial z}
\end{bmatrix} = \begin{bmatrix}
\frac{\partial u}{\partial x} & \frac{1}{2} \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \\
\frac{\partial w}{\partial x} & \frac{1}{2} \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right)
\end{bmatrix} + \begin{bmatrix}
0 & \frac{1}{2} \left( \frac{\partial u}{\partial x} - \frac{\partial w}{\partial z} \right) \\
0 & -\frac{1}{2} \left( \frac{\partial u}{\partial x} - \frac{\partial w}{\partial z} \right)
\end{bmatrix}$$

$$= \begin{bmatrix}
S_{xx} & S_{xz} \\
S_{zx} & S_{zz}
\end{bmatrix} + \begin{bmatrix}
0 & \frac{1}{2} \omega_y \\
-\frac{1}{2} \omega_y & 0
\end{bmatrix} = S_{yy} + \frac{1}{2} \epsilon_{iy} \omega_y = S_{jj} + W_{ij}$$

where $i, j = 1, 2; \epsilon_{ij} = \begin{cases} 0 & i = j \\ 1 & i < j \\ -1 & i > j \end{cases}$

Let $a = \frac{1}{2} S_{ij} S_{ji}$ and $b = \frac{1}{4} \omega_y^2$.

Define $\hat{\Omega} = \frac{b}{a + b + \epsilon^2}$; then $\hat{\Omega} = 1$ if flow is purely rotational and $\hat{\Omega} \ll 1$ if flow is irrotational but even vorticity is very large.

If we want to exclude some weak vortex, we can setup a reference $Q_{ref}$

Let $\Omega = \beta \hat{\Omega}$ where $\beta = \frac{1}{|Q|} \frac{|Q|}{|Q| + Q_{ref}}$. Pick $Q_{ref} = 1.0 E^{-4}$ and $\Omega = 0.5$, we get same vortex structure for a transitional boundary layer as what we used $\lambda_2$ iso-service. By using $\beta$, we can get very similar vortex
structure as using $\lambda_2$ iso-surface. Note that we really do not know what $\lambda_2 = -0.0001, -0.002, -0.006$ means and they are case-related. Q-criteria has same problem with the threshold which is also case-related. However, we do know $\Omega = 0.5$ means rotating core is formed and $\Omega$ is not case-related and has clear physical meaning. If $\beta = 1$ we obtain more vortex structures in upper-level of the Lambda vortex, which we call “Cloud” (See Figures 4-6).

Anyway, both $\lambda_2$ and Q-criteria do not have definite physical meaning, but $\Omega$ has clear physical meaning that if $\Omega > 0.5$, we think flow becomes rotational.

The 3-D analysis is exactly same as the above and the 3-D velocity decomposition can be written as
\[
dV' = \nabla V \cdot d\vec{l} = S \cdot d\vec{l} - d\vec{l} \times \frac{1}{2} \nabla \times \vec{V} = S \cdot d\vec{l} - d\vec{l} \times \frac{1}{2} \left[ \Omega (\nabla \times \vec{V}) + (1 - \Omega) \nabla \times \vec{V} \right] \]

$\Omega$ is the ratio of rotation, $\Omega = 1$ represents pure rotation and $\Omega = 0$ represents no rotation. $\Omega \cdot \frac{1}{2} \nabla \times \vec{V}$ represents the rotational vorticity

For 2-D, $S = \left[ \frac{\partial u}{\partial x} \left( \frac{1}{2} \frac{\partial u}{\partial z} \right) \right]$, $\nabla \times \vec{V} = j \left( \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right) d\vec{l} = dx \hat{i} + dz \hat{k}$

Note that the vortex ring is rotating with very high angle speed. The dimensional rotation speed for our case can be obtained as follows:
Characteristic length is L=0.004m, velocity V=340 m/s,
time $T = L/V = 0.004/340$ second = 0.0000117647 s, angle speed $\omega_\theta^d = \frac{1}{T} = 85,000$ 1/s and the rotating speed: $\omega_\theta = \frac{1}{2 \pi T} = 85,000 / (2 \pi)$ circles/s = 13,528 circles/second

and the ring rotating speed is: $\omega \cdot \omega_\theta = 4 \times 13528.569$ circles/s = 54,114 circles/s. This example shows the vortex ring has an extremely high rotation speed in turbulent flow. There is a misunderstanding that the flow could not rotate so fast since the viscosity exists. Actually, the pure rotation has no deformation and then has no dissipation. Therefore, the pure rotation is the least energy consumption state and thus most stable state.
3.5 Definition of shear

Shear is usually defined as 1-D or 2-D sharp velocity gradient, like Blasius solution which has shear: \( \frac{\partial u}{\partial z} \)

where \( u = u(Z) \). Some people use shear as deformation. If we use shear as deformation, shear cannot transfer to rotation directly since shear is not part of vorticity. However, most people include both deformation and non-rotational vorticity as “shear”. In this point of view, part of shear is non-rotational vorticity, which can transfer to rotational vorticity in flow transition. In other words, we can still say the process of flow transition is a process that “shear transfers to rotation.” For a 2-D boundary layer, it is easy to do the decomposition for the shear:

\[
\frac{\partial u}{\partial z} = \frac{1}{2} (\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}) + \frac{1}{2} (\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}) = \varepsilon_x + \omega_y
\]
The left part is deformation and the right part is vorticity. Note that the vorticity is not same as rotation.

Let us look at the 2-D boundary layer case:

1) If we have no perturbation, i.e. \( w = 0 \) and thus \( \frac{\partial w}{\partial x} = 0 \), the shear can be decomposed as half deformation and half vorticity without rotation. Clearly, vorticity does not mean rotation.

2) If we introduce a strong perturbation: \( \frac{\partial w}{\partial x} = -\frac{\partial u}{\partial z} \), then we have \( \frac{\partial u}{\partial z} = 2\omega \) without deformation and the shear becomes pure rotation. In such a case, vorticity is pure rotation. Or we can say the shear transfers to rotation. More exactly, we should say the irrotational vorticity transfers to rotational vorticity, which we believe is the nature of flow transition from the laminar state to turbulent state. Of course, it can only happen after the vorticity rollup from the wall surface.

3) If we introduce a strong perturbation: \( \frac{\partial w}{\partial x} = \frac{\partial u}{\partial z} \), then we have \( \frac{\partial u}{\partial z} = 2\varepsilon \) without vorticity and the shear becomes pure deformation.

Therefore, we should decompose vorticity to two parts: rotational vorticity like rigid body rotation and irrotational vorticity like laminar boundary layer.

### 3.6 Vortex breakdown

Some people argue that we do not have exact definition for vortex. Actually, if we define vortex as a vortex tube, vortex tube cannot break down. If we define vortex is a rotation core, the core is very stable and cannot break down either like tornado. However, if we have no definition for vortex, we then have no serious scientific research on turbulence. The definition of vortex is apparently a congregation of vortex lines with a rotation core. In practice, vortex breakdown is mainly caused by inappropriate pickup of Lambda 2 value of iso-surface.

Many people misunderstand vorticity as rotation, vortex as vortex tube, and vorticity line as vortex line. Actually, as addressed above, vortex is a rotating core which consists of vortex lines with leaking, but vortex tube is a tube with vorticity lines without leakage according to the definition in Davidson’s book. Therefore, vorticity does not mean rotation, vortex is not vortex tube, and vorticity line is not vortex line. On the other hand, vortex line is part of vorticity lines and rotating vorticity is part of vorticity.

### IV. Observation of vortex development from shear

#### 4.1 Vorticity rollup from the wall

It is widely accepted that the linear unstable modes play an important role in flow transition. The growth of the unstable T-S wave modes (both 2D and 3D) cause the linear instability of flow and then change to the non-linear instability stage. Following this idea, if we give the same linear unstable modes at the inflow, the DNS solution and the corresponding analytic linear solution should be same at the very beginning of the linear stage. People in general believe the turbulence is generated by these linear modes growth, non-linear interaction and breakdown through either abstract or convective instability. However, that is not the case.

We studied flow over a flat plate. The 2D laminar solution without inflow perturbation is the Blasius solution. After we add some 2D and 3D T-S waves at the inflow, the analytic linear solution can be written as:

\[
\hat{u} = \hat{u} + u'_{2D} + u'_{3D} = \hat{u} + a_{2D} \varphi_{2D}(z)e^{-i(\omega \tau - \alpha x)} + a_{3D} \varphi_{3D}(z)e^{-i(\omega \tau - \alpha x - \beta y)} + CC,
\]

where \( \hat{u} \) is Blasius solution, \( u'_{2D} \) and \( u'_{3D} \) are 2D and 3D perturbation (here T-S waves). We pick the magnitude of perturbation \( a_{2D} = 0.03 \) and \( a_{3D} = 0.01 \). \( \varphi_{2D}(z) \) and \( \varphi_{3D}(z) \) are 2D and 3D T-S modes respectively. \( \omega \) is real, \( \alpha \) and \( \beta \) are wave numbers in the streamwise direction \( x \) and spanwise direction \( y \) respectively. Both \( \alpha \) and \( \beta \) are complex numbers and CC is conjugate part of the perturbation. All of the parameters in the perturbation are same with our DNS.
Thus, using the analytic linear solution, we can get a time dependent perturbation growth and compare them with our DNS solution. We expected that when the magnitude of the 2D and 3D T-S waves is small, the analytical linear solution and DNS results are comparable. However, surprisingly in Fig. 4, we can see that the two solutions quickly depart from each other when the perturbation magnitude is still very small.

Figure 7 shows the distribution of spanwise vorticity on the central plane for both DNS solution and analytic linear solution from T-S modes. In the DNS solution, we can clearly see that the vorticity on the bottom were lifted and rolled up. The rolling up vorticity aggregated, generated the spanwise vortex (see Figure 8). In the analytic linear solution, the bottom vorticity were also lifted up very little, since the largest vortices are away from the wall in Figure 7(b). We know that for the Blasius solution, the spanwise vorticity $\omega_y = \frac{\partial u}{\partial x} - \frac{\partial w}{\partial x} = \frac{\partial u}{\partial x}$, since $w = 0$. Obviously, $\frac{\partial u}{\partial x}$ have the maximum value near the wall surface. Thus, in the analytic linear solution, the bottom vorticity were also away from the wall surface a little. T-S waves contributed the normal direction velocity $w'$, which lifts up the bottom vorticity. However, we cannot see the ‘rollup’ vorticity like the DNS solution in our analytic linear solution.

Figure 8 shows the distribution of spanwise vorticity on the central plane for both DNS solution and linear solution when $\Lambda$-vortex was just formed at $t=6.3T$. Here T is the period of T-S wave. There are some $\lambda_2$ structures in DNS solution. And it can be found that the $\lambda_2$ structure is deformed when the spanwise vorticity is gathered into its center. However, we can find nothing from the linear solution. It shows the same pattern of spanwise vorticity distribution as the original structure. No $\lambda_2$ structure and $\Lambda$-vortex are observed in the linear case.
Figure 8. Spanwise vorticity on the central plane with $\lambda_2$ vortical structures (a) DNS results, (b) analytic linear solution

Compare Fig. 7 and Fig. 8, it is clearly that $\Lambda$-vortex can appear only after the vorticity rollup from the wall. Linear unstable modes contributes to generate $w'$ to lift up the spanwise vorticity. In DNS solution, once the vorticity left the wall surface, the trend from fluid shear to fluid rotation will happen. However, in analytic linear solution, the growing vorticity cannot roll up since it is linear, and thus has no $\Lambda$-vortex appeared.

Now we can give our conclusion that the role of linear unstable modes is only to generate the normal direction velocity $w'$, roll up the spanwise vorticity to go inside the flow. Linear unstable modes cannot directly cause the formation of spanwise vortex, and $\Lambda$-vortex. Actually the formation of vortex is because of that the shear must transform to rotation, which is the inherent property of flow. This statement has been explained in our previous paper.

Figure 9. The central plane at $t = 5.8T$ contoured by (a) spanwise vorticity (b) $\frac{\partial u}{\partial z}$
Fig. 9 shows the early vorticity rollup from the wall at the early stage. The red color represents the high spanwise vorticity and it is easy to find the spanwise vorticity is dominated by $\frac{\partial u}{\partial z}$ since the vorticity is mainly shear (irrotational vorticity) and strong rotation has not been developed. However, in Fig. 10 at late time steps, the shear is broken (K-H type, see Fig. 10 (a)) and a rotational core is developed. The process could be considered as the shear transfers to rotation or irrotational vorticity becomes rotational vorticity. However, $-\frac{\partial w}{\partial x}$ is still very small (0.045) and the spanwise vorticity (1.6) is dominated by $\frac{\partial u}{\partial z}$.

4.2 Shear to rotation transformation

From Fig. 11, it can be found that the high shear layer rolls up, but no rotation cores or vortex rings can be found at $t = 5.8T$. However, when $t = 6.2T$, the high shear layer breaks up and two rotation cores
(vortex rings) appear one after the other. This shows the shear is unstable and must transfer to rotation as a vortex ring when the shear is seriously stretched.

4.3 Fast rotation core formation

After the vortex rings form, the development of vorticity, $\frac{\partial u}{\partial x}$ and $\frac{\partial w}{\partial x}$ in the central section of the first vortex ring was traced at several consequent time steps, $t = 6.2T, 6.5T, 6.8T$ (See Fig. 13-15). It can be seen that two rings, three rings and four rings appear and become matured. For this pseudo-two dimensional problem, the rotation strength is directly related to the growth of $dw/dz$ as the main stream is in the x-direction. Table 2 gives the velocity gradients at the first vortex core at successive times. The first ring-like vortex is chosen because it appears first, and can represent the developing process of other vortex rings. At $t = 6.20T$ (Figure 10), $\frac{\partial u}{\partial z} \approx 1.88$ while $\frac{\partial w}{\partial x} \approx -0.024$, the rotation is still weak. After that as time is going, rotation become stronger. At $t = 6.50T$ (Figure 11), $\frac{\partial u}{\partial z} \approx 1.26$ while $\frac{\partial w}{\partial x} \approx -0.131$. At $t = 6.80T$ (Figure 15) $\frac{\partial u}{\partial z} \approx 0.34$ while $\frac{\partial w}{\partial x} \approx -0.133$. Although the spanwise vorticity $\omega_y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}$ is decreasing, the part of vorticity account for vortex rotation, $2.0 \times |\frac{\partial w}{\partial x}|$ is growing. The decreasing of spanwise vorticity comes from the decreasing of $\frac{\partial u}{\partial z}$, which seems to be a result as the vortex ring moves away from the wall. The first vortex ring which we are interested is located at $z = 2.4, 3.4, 4.5$ at $t = 4.46T, 4.68T, 4.90T$ respectively. Clearly, the rotation at beginning is zero, but quickly become large as partially rotating and finally fast rotation, which qualitatively agrees with Cai’s experiment. In the pseudo-two dimensional case, the vorticity does not change much, but $|dw/dx|$ growth is fast and the rotation become strong which is dominated by the growth of $dw/dx$.

Figure 13. Distribution of $du/dz$ and $dw/dx$ in two vortex rings ($t = 6.20T$)
Figure 14. Distribution of $\frac{du}{dz}$ and $\frac{dw}{dx}$ in two vortex rings ($t = 6.50T$)

Figure 15. Distribution of $\frac{du}{dz}$ and $\frac{dw}{dx}$ in the center section of four vortex rings ($t = 6.8T$)
Table 2. The pseudo 2D velocity gradients of the first ring-like vortex at successive times

<table>
<thead>
<tr>
<th>t = 6.2T</th>
<th>t = 6.5T</th>
<th>t = 6.8T</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \nabla U = \begin{bmatrix} \frac{\partial u}{\partial x} &amp; \frac{\partial u}{\partial z} \ \frac{\partial w}{\partial x} &amp; \frac{\partial w}{\partial z} \end{bmatrix} )</td>
<td>[\begin{bmatrix} -0.004891 &amp; 1.878912 \ -0.023778 &amp; -0.122260 \end{bmatrix}]</td>
<td>[\begin{bmatrix} 0.179842 &amp; 1.264818 \ -0.131329 &amp; -0.275229 \end{bmatrix}]</td>
</tr>
</tbody>
</table>

4.4 Distribution of \( \frac{du}{dz} \) and \( \frac{dw}{dx} \) in the vortex central plane

\[\omega_y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = \frac{\partial u_i}{\partial z} + \frac{\partial u_i}{\partial x} + \frac{\partial w_i}{\partial z} - \frac{\partial w_i}{\partial x} = \frac{1}{2} \left[ \frac{\partial u_i}{\partial z} + \frac{\partial w_i}{\partial x} \right] + \frac{1}{2} \left[ \frac{\partial u_i}{\partial z} - \frac{\partial w_i}{\partial x} \right] + \frac{\partial u_i}{\partial z} - \frac{\partial w_i}{\partial x} \]

\[= \varepsilon_{xc} + \omega_{irrot} + \omega_{rot} \]

where, \( u = u_i + u_2, \frac{\partial u_i}{\partial z} \approx -\frac{\partial w_i}{\partial x}, \) \( w \) is internal, but \( w_i \) is external

\( \varepsilon_{xc} \) is deformation, \( \omega_{irrot} \) is irrotational, \( \omega_{rot} \) is rotational vorticity

Note that, at beginning, \( |\frac{\partial u}{\partial z}| > > |\frac{\partial w}{\partial x}| \) and \( u_2 << u_1 = u \). As \( |\frac{\partial w}{\partial x}| \) grows, the deformation will become smaller and the rotation becomes strong. At that moment \( u \\approx u_2 >> u_1 \), the fluid particle is dominated by rotation.

V. Conclusion

Although this is a simple analysis for the central section of a vortex ring, we can still found that
1) Vorticity rolls up from the wall due to the unstable T-S modes
2) The vorticity rollup will form a shear layer which is strongly stretched
3) The fluid cannot tolerate the high shear and shear layer breaks and the rotation core (vortex) is formed.
4) The motion of a fluid particle should be decomposed as four parts instead of three: translation, deformation, irrotating vorticity and rotating vorticity. In the vortex development process, the irritating vorticity will become rotating vorticity and the extremely high rotating speed will be formed (fast rotation cores). In this way, the deformation part will be reduced and the energy consumption will kept minimum
5) In other words, shear which is dominant in laminar flow is an unstable state and the fast rotation core which is dominant is turbulent flow is a stable state.
6) A 3-D analysis on irrotating vorticity transfer to rotating vorticity will be given in the final paper.

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