A Definition of Vortex Vector and Vortex

Shuling Tian
Yisheng Gao
Xiangrui Dong
Chaoqun Liu

Technical Report 2017-06

http://www.uta.edu/math/preprint/
A Definition of Vortex Vector and Vortex

Shuling Tian¹,², Yisheng Gao², Xiangrui Dong²,³, Chaoqun Liu²*

¹College of Aerospace Engineering, Nanjing University of Aeronautics and Astronautics, Nanjing, Jiangsu, 210016, China
²Department of Mathematics, University of Texas at Arlington, Arlington, Texas, USA
³National Key Laboratory of Transient Physics, Nanjing University of Science & Technology, Nanjing, Jiangsu, 210094, China

Vortex is ubiquitous in nature. However, there is not a consensus on the vortex definition in fluid dynamics. Lack of mathematical definition has caused considerable confusions in visualizing and understanding the coherent vortical structures in turbulence. According to previous study, it is realized that vortex is not the vorticity tube and vorticity should be decomposed into a rotational part which is the vortex vector and a non-rotational part which is the shear. In this paper, several new concepts such as fluid rotation of local point, the direction of fluid rotation axis and the strength of fluid rotation are proposed by investigating the kinematics of fluid element in the 2D and 3D flows. A definition of a new vector quantity called vortex vector is proposed to describe the local fluid rotation. The direction of the vortex vector is defined as the direction of local fluid rotation axis. The velocity components in the plane orthogonal to the vortex vector have zero derivatives along the vortex vector direction. The magnitude of the vortex vector is defined as the rotational part of vorticity in the direction of the vortex vector, which is the twice of the minimum angular velocity of fluid around the point among all azimuth in the plane perpendicular to vortex vector. According to the definition of the vortex vector, vortex is defined as a connected flow region where the magnitude of the vortex vector at each point is larger than zero. The new definition for the vortex vector and vortex follows three principles: 1. Local in quantity, 2. Galilean invariant, 3. Unique. The definitions are carefully checked by DNS and LES examples which clearly show that the new defined vortex vector and vortex can fully represent the complex structures of vortices in turbulence.

I. INTRODUCTION

Vortex is a special existence form of the fluid motion featured as rotation of fluid elements and is ubiquitous in the nature. Vortex can be observed in many organized flow structures ranged from hurricanes to tornadoes, from airplane trailing vortices to swirling flows in turbines, and from vortex rings at the exit of a pipe to coherent structures in turbulent boundary layer flow. Although intuitively a vortex can be easily recognized, it is surprisingly difficult to give an unambiguous definition for vortex¹. As early as in 1858, Helmholtz proposed the concepts of vortex lines and vortex filaments to investigate the simplest vortex motion.² However, so

* Corresponding author: Email cliu@uta.edu
far the problem of the mathematical definition of vortex still remains an open issue in fluid dynamics.\textsuperscript{1} The lack of a consensus on the vortex definition has caused considerable confusions in visualizing and understanding the vortical structures, their evolution, and the interaction in complex vortical flows, especially in turbulence.\textsuperscript{3,4}

As one of important physical quantities in the fluid dynamics, vorticity is mathematically defined as the curl of velocity vector filed, but has no very clear physical meaning. In classic vorticity dynamics, vorticity is always interpreted as twice the angular velocity of the instantaneous principal axes of the strain-rate tensor of a fluid element. If $\nabla \times \vec{v} = 0$ at every point in a flow, the flow is called irrotational otherwise called rotational, which implies that the fluid elements have a finite angular velocity. Therefore, people always qualitatively identify a vortex as a connected fluid region with relatively high concentration of vorticity and treat vortex as vorticity line or vorticity tube. This may be intuitively reasonable for the large scale vortical structures like tornadoes to equalize vortex and vorticity in early era when people assumed flow is inviscid. However, in 3D and viscous flow vortex cannot be represented by vorticity tube in general. An immediate counter-example is 2D Blasius laminar boundary layer where vorticity is very large near the wall surface, but no flow rotation or vortex is found.\textsuperscript{5} In addition, vortices are not necessarily the concentration of vorticity in turbulent flow. Based on DNS results for late flow transition, Wang et al\textsuperscript{6} found that vorticity will be reduced when the vorticity lines enter the vortex region and the vorticity magnitude inside vortex is much smaller than the surrounding area, especially near the solid wall. In complex vortical flow, especially in turbulence, the vortical structures cannot be represented by vorticity lines or tubes and the direction of vortex axis is also always different from the direction of vorticity lines or tubes. On the other hand, the existence of a vortex tube does not mean the existence of a vortex, for example, a self-closed ring-like “vorticity tube” exists before the hairpin vortices, but no vortex can be identified by any vortex identification criterion.\textsuperscript{6} So, vorticity does not imply the rotation of fluid element and it is not adequate to reveal all types of vortical structures. In general, vortex is a physical phenomenon in nature, but vorticity is a mathematical definition of velocity curl and there is no reason to say vortex can be represented by or equivalent to vorticity tube or vorticity surface.

Researchers have recognized that the vortical structures play an important role in turbulence which has been found to be dominated by spatially coherent and temporally evolving vortical motions, which is popularly called coherent structures. A famous comment is given by Küchemann\textsuperscript{7}: “Vortices are the sinews and muscles of turbulence.” While in many studies of classic vortex dynamics, many qualitative definitions for vortex are developed, a rigorous quantitative mathematical definition becomes crucial nowadays in understanding vortex structures, their evolution and interactions in turbulence. As Jeong and Hussain\textsuperscript{8} stressed, when defining the vortex, the quantitative criteria should satisfy the need for Galilean-Invariance. In recent decades, many vortex identification methods were proposed, mainly including Eulerian method and Lagrangian method\textsuperscript{9,10,11}. The $\Delta$-criterion\textsuperscript{12-14}, $Q$-
criterion\textsuperscript{15}, \(\lambda_{1}\)-criterion\textsuperscript{16,17}, \(\lambda_{2}\)-criterion\textsuperscript{8}, \(\Omega\)-criterion\textsuperscript{18} and the most recent \(\lambda_{\omega}\)-criterion\textsuperscript{19} are representative among many of the most common Eulerian local vortex identification methods which are based on the local point-wise analysis of the velocity gradient tensor\textsuperscript{20}. Based on these methods, a vortex exits in the connected region where the criterion is met. Epps gave a very detailed review of vortex identification methods in Ref. \textsuperscript{10} and showed that despite the vast number of criterions have been proposed, the field seems to lack an impartial way to determine which criterion is best. However, though it is hard to judge which method is superior over others, it can be found that all these criterions are described by scalar variable but are not able to give the direction and the exact strength of a vortex which has both direction and strength. Therefore, a precise mathematical definition for vortex is still an open question.

Since the intuitive observation of vortex shows that the vortex has a rotational axis, a natural idea for developing the possible criterion of vortex definition is to find a vector from the decomposition of velocity gradient tensor. As a vector, the vorticity, which is the curl of velocity, is the most straightforward way to be used for the purpose of identifying vortex, but as mentioned above, it cannot distinguish a vortical region from a shear layer, Blasius solution for example. In order to remove shear from the velocity gradient tensor, Kolář\textsuperscript{21} proposed a triple decomposition method from which a residual vorticity can be found to represent the pure rigid-body rotation of fluid element. However, the triple decomposition is not unique and a basic reference frame has to be used to solve this problem. Searching for the Euler angles that define the basic reference frame is an expensive optimization problem. In Ref. \textsuperscript{22}, Kolář et al introduced the concept of average co-rotation of material line segments near a point and applied the average co-rotation to vortex identification. Numerical results\textsuperscript{22,23} showed that the method can accurately identify the complex vortical structure, such as hairpin vortex in a turbulent boundary layer flow. However, the averaged co-rotation vector is evaluated by an integral equation and it will be difficult to study the transport of vortex in turbulence by the quantity. Thus, finding a rigorous mathematical definition of vortex is still an open question.

Though vorticity does not always represent rotation of fluid element, Liu et al\textsuperscript{18} pointed out that vorticity \(\vec{\xi} = \nabla \times \vec{V}\) should be further decomposed into two parts: one is the rotational part \(\vec{R}\) contributed to rotation and the other one \(\vec{S}\) is non-rotational part (shear) like the vorticity without rotation in laminar boundary layer flows. However, they didn’t show how to decompose the vorticity and did not give formula of \(\vec{R}\) and \(\vec{S}\). In this paper, based on the physical meaning of vortex, several new concepts such as fluid rotation of local point, the direction of fluid rotation axis and the strength of fluid rotation are proposed by investigating the kinematics of fluid element in 2D and 3D flows. A mathematical definition for vortex vector including the direction and magnitude is presented to describe the fluid rotation in the current paper. The direction of vortex vector is the direction of local fluid rotation axis which is determined by the character that the velocity components in the plane normal to the direction of the vortex vector do not change
along the direction of the local fluid rotation axis, in the other words, these velocity components orthogonal to the vortex vector have zero derivatives in the vortex vector direction. This leads to the definition of the vortex vector direction. The magnitude of the vortex vector is measured by the rotational part of vorticity in the direction of the vortex vector. According to the mathematical definition of the vortex vector, vortex is defined as a connected fluid region where the magnitude of the vortex vector at each point is larger than zero.

The paper is organized as follows. The physical meaning of vortex, the method determining the direction of local rotation axis and the strength of fluid rotation are given by the analysis on the motion of the 2D/3D fluid element in Section 2 and the mathematical definition of vortex vector and vortex are given in Section 3. The application of the new mathematical definitions on a number of computational results obtained from DNS and LES are presented in Section 4 and the conclusion remarks are given in last section.

V. Conclusions

Based on investigating the kinematics of fluid element in the 2D/3D flows, a definition of a new vector quantity called vortex vector is given in this paper to describe vortex. Three principles are followed during the definition of vortex vector: 1. The definition must be a local quantity as tens of thousands of vortices exist in turbulence and global or group quantity is not viable to define them; 2. The definition must be Galilean invariant; 3. The definition must be unique. Different from other vortex identification methods which are all scalar criterions, the newly defined variable is a vector quantity with a unique, Galilean invariant direction and a unique magnitude.

The vortex vector represents the local fluid rotation. The direction of the vortex vector is defined as the local fluid rotation axis along which the velocity components in the plane orthogonal to the vortex vector have zero derivatives in the vortex vector direction or \( \frac{\partial u}{\partial z} = 0 \) and \( \frac{\partial v}{\partial z} = 0 \) when the axis is set as parallel to the Z axis. The rotation axis is unique and Galilean invariant. The magnitude of the vortex vector is defined as the rotational part of vorticity in the direction of the vortex vector, which is twice of the minimum angular velocity of fluid around the point among all azimuth in the plane perpendicular to vortex vector and is also unique. Based on the definition of vortex vector, vortex is defined as a connected flow region where the magnitude of the vortex vector at each point is larger than zero. According to the definition of the vortex vector and vortex, the concepts of vortex line and vortex tube are also proposed to demonstrate the vortex structures.

The definitions of vortex vector and vector are evaluated using 2D/3D DNS/LES databases and are compared with vorticity and other vortex identification methods, such as Q-criterion and \( \Omega \)-criterion. The results show that, unlike vorticity lines or tubes, the vortex lines or tubes based on the newly defined vortex vector are highly consistent with the cores of vortex which can be accurately represented by the iso-surface of the magnitude of vortex vector. Vortex vector can fully describe the vortical structures...
in not only incompressible flow but also compressible flow since it is derived only based on the kinematics of fluid element, but not fluid dynamics. Different from most vortex identification methods which only have a scalar variable and use the iso-surface to demonstrate the vortex structure, vortex vector, as a vector variable, can not only represent the accurate rotational strength of vortex but also can show the precise rotational direction, which is the advantage that most of vortex identification methods do not have.

As a rigorous mathematical definition with direction and magnitude, the newly defined vortex vector and vortex can be used to quantitatively investigate the physics of the generation and sustenance of vertical structures in turbulence. In the further work, we will study the transport equation for vortex vector and investigate the mechanics of the vortex generation and evolution in turbulence.

ACKNOWLEDGMENTS

The authors are grateful to Texas Advanced Computing Center (TACC) for providing computation hours. The work was funded by the Priority Academic Program Development of Jiangsu Higher Education Institutions. This research was also supported in part by MURI FA9559-16-1-0364. This work is accomplished by using Code DNSUTA which was released by Dr. Chaoqun Liu at University of Texas at Arlington in 2009.

17 J. Zhou, R. J. Adrian, S. Balachandar and T. M. Kendall, “Mechanisms for generating coherent packets of hairpin vortices in channel flow,” J. Fluid Mech. 387,
353(1999).


