QUADRATIC ALGEBRAS ASSOCIATED WITH THE UNION OF A QUADRIC AND A LINE IN $\mathbb{P}^3$

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Abstract. We define a family of graded quadratic algebras A_σ (on 4 generators) depending on a fixed nonsingular quadric Q in $\mathbb{P}^3$, a fixed line L in $\mathbb{P}^3$ and an automorphism $\sigma \in \text{Aut}(Q \cup L)$. This family contains $\mathcal{O}_{\text{II}}(\mathcal{M}_{\in}(\mathbb{C}))$, the coordinate ring of quantum 2×2 matrices. Many of the algebraic properties of A_σ are shown to be determined by the geometric properties of $\{Q \cup L, \sigma\}$. For instance, when $A_\sigma = \mathcal{O}_{\text{II}}(\mathcal{M}_{\in}(\mathbb{C}))$, then the quantum determinant is the unique (up to a scalar multiple) homogeneous element of degree 2 in $\mathcal{O}_{\text{II}}(\mathcal{M}_{\in}(\mathbb{C}))$ that vanishes on the graph in $\mathbb{P}^3 \times \mathbb{P}^3$ of $\sigma|_Q$ but not on the graph of $\sigma|_L$. Following [ATV1, 2], we study point and line modules over the algebras A_σ, and find that their algebraic properties are consequences of the geometric data. In particular, the point modules are in one-to-one correspondence with the points of $Q \cup L$, and the line modules are in bijection with the lines in $\mathbb{P}^3$ that either lie on Q or meet L. In the case of $\mathcal{O}_{\text{II}}(\mathcal{M}_{\in}(\mathbb{C}))$, when q is not a root of unity, the quantum determinant annihilates all the line modules $M(\ell)$ corresponding to lines $\ell \subset Q$; the determinant generates the whole annihilator for such $\ell \subset Q$ if and only if $\ell \cap L = \emptyset$.

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The purpose of this paper is to broaden the class of examples hitherto discussed in relation to the non-commutative geometry developed in [ATV1, 2]. The main focus in [ATV1, 2] was on regular algebras of dimension 3, for which many algebraic properties were shown to be a consequence of certain geometric data; namely, a scheme E (typically a cubic divisor in $\mathbb{P}^2$) and an automorphism of E. This geometric theme is continued in [SSt] and [LS] in their study of the Sklyanin algebra, a 4-dimensional regular algebra, which is related to an elliptic curve in $\mathbb{P}^3$ and an automorphism in a similar way. We assume the reader is familiar (to some extent) with [ATV1, 2] and [LS].

This paper describes a family of regular algebras of dimension 4 defined in terms of a variety $Q \cup L$ in $\mathbb{P}^3$, where $Q \subset \mathbb{P}^3$ is a nonsingular quadric and $L \subset \mathbb{P}^3$ is a line (in general position), together with an automorphism σ of $Q \cup L$. [SSt] and [LS] show that many of the techniques developed in [ATV1, 2] for the 3-dimensional regular algebras carry over in dimension 4 to the Sklyanin algebra – our goal is to show these techniques also carry over to the algebras associated to $\{Q \cup L, \sigma\}$.

Our class of algebras splits into two types: those for which σ preserves the rulings on Q and those for which σ interchanges them. Examples of the first kind (presented in §1) are $\mathcal{O}_{\rm II}(\mathcal{M}_{\in}(\mathbb{C}))$, the coordinate ring of quantum 2×2 matrices, and $\mathcal{O}_{\rm II}(\mathfrak{sp}\mathbb{C}^4)$, the coordinate ring of quantum symplectic 4-dimensional space. Moreover, families of algebras on n^2 generators which contain $\mathcal{O}_{\rm II}(\mathcal{M}_{\backslash}(\mathbb{C}))$ are defined in [AST], [Su, §4b] and [T] respectively, and when $n = 2$, most of their algebras are included in our family (with $\sigma|_L =$ identity).

Section 1 begins with the geometric definition of the algebras A. We then describe A in terms of generators and relations. In particular, we show they are skew-polynomial rings which are noetherian domains of global homological dimension 4 having the same Hilbert series as the polynomial ring in 4 variables. The $\mathbb{P}^3$ containing $Q \cup L$ is naturally identified with $\mathbb{P}(A_1^*)$, so the elements of A_1 are linear forms on $Q \cup L$, and elements of $A_1 \otimes A_1$ are bilinear forms on $\mathbb{P}^3 \times \mathbb{P}^3$. We show that any element $\omega \in A_1$ that is an eigenvector for $\sigma|_Q$ (i.e., $\omega \circ \sigma|_Q$ is a scalar multiple of ω) and vanishes on L is a normal element of A – there are at least two (independent) such elements. Moreover, there is a unique (up to a scalar multiple) element Ω in A_2 that vanishes on the graph in $\mathbb{P}^3 \times \mathbb{P}^3$ of $\sigma|_Q$ but not on the graph of $\sigma|_L$ – in addition to this geometric property, Ω is a normal element of A.

In §2 the point modules are shown to be in bijection with the points of $Q \cup L$, and the line modules in natural bijection with the lines in $\mathbb{P}^3$ that either lie on Q or meet the line L. If M is either a point or a line module over A then $M \cong A/AW$ where W is the linear subspace of A_1 vanishing on the respective point or line.

The quotient algebra $A/A\Omega$ is shown in §3 to be a domain. Here the techniques employed from [ATV1, 2] analyse $A/A\Omega$ in terms of the geometric data $\{Q, \sigma|_Q, \mathcal{L}\}$ where $\mathcal{L}$ is the invertible sheaf of linear forms on Q. The key idea is to construct the algebra $B = \bigoplus_{n \geq 0} H^0(Q, \mathcal{L} \otimes \mathcal{L}^\sigma \otimes \cdots \otimes \mathcal{L}^{\sigma^{\backslash - \infty}})$, where $\mathcal{L}^\sigma$ is the pullback of $\mathcal{L}$ by $\sigma|_Q$, and to show that $B \cong A/A\Omega$. The algebra B (and hence $A/A\Omega$) is analogous to the homogeneous coordinate ring of Q, which is isomorphic to $\bigoplus_{n \geq 0} H^0(Q, \mathcal{L}^\backslash)$. In the language of [AV], B is a twisted homogeneous coordinate ring. In this way, we can view Ω as playing a role similar to that of the defining equation of Q.

We investigate the annihilators of point and line modules in §4, proving that when σ has infinite orbit at "most" points of Q then $A\Omega$ is the annihilator of the line modules

$M(\ell)$ where ℓ is a line on Q such that $\ell \cap L = \emptyset$. We end the section by restricting to the algebras for which σ preserves the rulings on Q. In particular, we show that when q is not a root of unity, then every line module over $\mathcal{O}_{\mathrm{II}}(\mathcal{M}_{\in}(\mathbb{C}))$ has nonzero annihilator. Moreover, every homogeneous prime ideal of $\mathcal{O}_{\mathrm{II}}(\mathfrak{sp}\mathbb{C}^4)$, when q is not a root of unity, is shown to be the annihilator of some linear module.

The kernel of a surjective map from a line module to a point module is observed in §2 to be a shifted line module. We devote the last section (§5) to identifying the line corresponding to this kernel.

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1. Definition of the Algebras

The algebras discussed in this paper are graded algebras on four generators (of degree one) defined by six homogeneous quadratic relations. We define the algebras in terms of a certain subvariety of $\mathbb{P}^3$, namely the union of a fixed quadric Q and a fixed line L, and an automorphism σ of that variety. We prove that the algebras so obtained are skew-polynomial rings, but it is the geometric description of the algebras that will be exploited throughout the paper.

Unless otherwise stated, we will work over an algebraically closed field k, where $\mathrm{char}(k) \neq 2$. We fix a nonsingular quadric $Q \subset \mathbb{P}^3$ and a line $L \subset \mathbb{P}^3$, $L \not\subset Q$, that meets Q in two distinct points. Certain restrictions (see (1)-(4) below) are placed on $\sigma \in \mathrm{Aut}(Q \cup L)$. These may seem artificial at first glance, but the remark following the definition of σ explains why these restrictions are natural. We define $\sigma \in \mathrm{Aut}(Q \cup L)$ subject to the following conditions:

(1) $\sigma(Q) = Q$ and $\sigma(L) = L$,
(2) $\sigma|_{Q \cap L}$ is the identity,
(3) $\sigma|_Q$ and $\sigma|_L$ are restrictions of linear automorphisms of $\mathbb{P}^3$, and
(4) σ is not the restriction of a linear automorphism of $\mathbb{P}^3$.

Remark . Conditions (1) and (3) are automatically satisfied by an automorphism of $Q \cup L$. However, if condition (4) is omitted, and σ does extend to $\mathbb{P}^3$, then the algebra obtained (see below) is a "twist" ([ATV2, §8]) of the commutative polynomial ring by an automorphism. Since the categories of graded modules over an algebra and its twist are equivalent ([ATV2, 8.5]), there is nothing new to say about such algebras.

If σ violates condition (2), then one obtains algebras which are not domains and therefore are not noetherian (Auslander-)regular algebras ([ATV2], [L]). As our main emphasis is on the techniques developed in [ATV1, 2] and [LS], this paper will not discuss such algebras, although they are probably straightforward to understand.

There exist choices of σ satisfying the above conditions that yield the coordinate ring of quantum 2×2 matrices and other quantum group examples (see 1.5-1.7).

The graph of σ will be denoted by Γ_σ; that is,

$$\Gamma_\sigma := \{(p, \sigma(p)) \in \mathbb{P}^3 \times \mathbb{P}^3 \ : \ p \in Q \cup L\}.$$

Similarly, $\Gamma_\sigma(Q)$ and $\Gamma_\sigma(L)$ will denote the graphs of $\sigma|_Q$ and $\sigma|_L$ respectively.

Let V be the 4-dimensional k-vector space of linear forms on the copy of $\mathbb{P}^3$ that contains $Q \cup L$, and let $T(V)$ be the tensor algebra on V. Thus, $\mathbb{P}^3 = \mathbb{P}(V^*)$ and $V \otimes V$ acts as bilinear forms on $\mathbb{P}(V^*) \times \mathbb{P}(V^*)$. Let I_σ be the subspace of $V \otimes V$ defined by

$$I_\sigma := \{f \in V \otimes V \ : \ f(\Gamma_\sigma) = 0\}.$$

Definition . With the above notation, we define a family of algebras $\{A_\sigma\}$ by

$$A_\sigma := \frac{T(V)}{\langle I_\sigma \rangle}.$$

We recall, for the benefit of the reader, certain properties of the quadric Q which will be used frequently.

Facts .

(a) *For each $x \in Q$ there are exactly two lines lying on Q which pass through x.*
(b) *Let ℓ_1, ℓ_2 be two distinct intersecting lines on Q. Then any other line on Q meets only one of ℓ_1, ℓ_2. This defines two families (rulings) of lines on Q.*
(c) *Every plane meets Q in either a nondegenerate conic or in two distinct lines.*
(d) *If a line $\ell \not\subset Q$ meets Q in two distinct points, then there exist precisely two planes containing ℓ and meeting Q in degenerate conics.* ■

Notation . Since L meets Q at two distinct points, these facts show there are exactly four "special", distinct (since $L \not\subset Q$) lines lying on Q that meet L. Let $e_1,\, e_2,\, e_3,\, e_4 \in Q$ denote the four distinct points at which these lines pairwise intersect, and set $\{e_2,\, e_3\} = Q \cap L$. Notice that $e_1, \dots, e_4$ span $\mathbb{P}^3$.

We define four linear forms $x_1, \dots, x_4$ on $\mathbb{P}^3$ by:

$$\mathcal{V}(x_i) = \text{ linear span } \{e_j \ : \ 1 \le j \le 4,\, j \ne i\} \quad \text{for } 1 \le i \le 4.$$

Thus, $\{x_1, \dots, x_4\}$ is a basis for V. Clearly, $L = \mathcal{V}(x_1,\, x_4)$. We aim to show that we can assume $Q = \mathcal{V}(x_1x_4 + x_2x_3)$.

Lemma 1.1. *The nonsingular quadrics which contain the four special lines on Q that meet L are precisely the $\mathcal{V}(\lambda_1 x_1 x_4 + \lambda_2 x_2 x_3)$ where $\lambda_1,\, \lambda_2 \in k \setminus \{0\}$. In particular, we can rescale x_i such that $Q = \mathcal{V}(x_1x_4 + x_2x_3)$.*

Proof. Write U for the union of the four special lines on Q that meet L, and denote the line passing through e_i and e_j by ℓ_{ij}.

We first consider the singular quadrics which contain U. Since $\ell_{12} \cap \ell_{34} = \emptyset = \ell_{13} \cap \ell_{24}$, U cannot lie on a rank 1 quadric (a plane). A rank 3 quadric contains a point through which every line on the quadric passes, so U cannot be contained in a rank 3 quadric. A rank 2 quadric, being the union of two planes, contains U if and only if two of the four lines lie in one of the planes and the other two in the other. Therefore, the lines determine the planes. It follows that $\mathcal{V}(x_1x_4)$ and $\mathcal{V}(x_2x_3)$ are the only rank 2 (and the only singular) quadrics containing U. Set $q_1 = x_1x_4$, $q_2 = x_2x_3$.

Suppose there exists a nonsingular quadratic form $q_3 \notin kq_1 \oplus kq_2$ such that $U \subset \mathcal{V}(q_3)$. Then we obtain a net $\mathcal{N}$ of quadrics which contain U, where

$$\mathcal{N} = \left\{ q_\lambda = \sum_{i=1}^{3} \lambda_i q_i \ : \ \lambda = (\lambda_1,\, \lambda_2,\, \lambda_3) \in \mathbb{P}^2 \right\} \simeq \mathbb{P}^2.$$

Let $M_i \in M_4(k)$, resp. $M_\lambda \in M_4(k)$, denote the matrix of the bilinear form corresponding to q_i, resp. q_λ. Then $M_\lambda = \sum \lambda_i M_i$. Now the vanishing of the determinant defines a degree 4 curve $\{\lambda \in \mathbb{P}^2 \ : \ \det M_\lambda = 0\}$ in $\mathbb{P}^2$ consisting of the singular quadrics in $\mathcal{N}$. This contradicts the fact that there exist only two singular quadrics which contain U. The result follows. ■

For the rest of the paper we assume $Q = \mathcal{V}(x_1x_4 + x_2x_3)$ and $L = \mathcal{V}(x_1, x_4)$.

Remark 1.2. By condition (2) defining σ, the points e_2 and e_3 are each fixed by σ. Therefore, since $\sigma|_Q$ is the restriction of a linear automorphism of $\mathbb{P}^3$, it follows that the two lines on Q through e_2, resp. e_3, are either fixed or interchanged by σ. Hence, the set $\{e_1, e_4\}$ is stable under σ, so that only two types of automorphism σ arise.

Lemma 1.3. *The automorphism σ either fixes each of the two rulings on Q or interchanges them.*

(*a*) *If σ fixes the two rulings on Q then $A_\sigma = k[x_1, x_2, x_3, x_4]$ with the six defining relations:*

$$x_2x_1 = \alpha x_1x_2, \qquad x_3x_1 = \lambda x_1x_3, \qquad x_4x_1 = \alpha\lambda x_1x_4,$$
$$x_4x_3 = \alpha x_3x_4, \qquad x_4x_2 = \lambda x_2x_4, \qquad x_3x_2 - \beta x_2x_3 = (\alpha\beta - \lambda)x_1x_4,$$

for some $\alpha, \beta, \lambda \in k \setminus \{0\}$ determined by σ, where $\lambda \neq \alpha\beta$.

(*b*) *If σ interchanges the two rulings on Q then $A_\sigma = k[x_1, x_2, x_3, x_4]$ with the six defining relations:*

$$x_3x_4 = \alpha x_1x_3, \qquad x_2x_4 = \lambda x_1x_2, \qquad x_4^2 = \alpha\lambda x_1^2,$$
$$x_4x_2 = \alpha x_2x_1, \qquad x_4x_3 = \lambda x_3x_1, \qquad \beta x_3x_2 - x_2x_3 = (\lambda - \alpha\beta)x_1^2,$$

for some $\alpha, \beta, \lambda \in k \setminus \{0\}$ determined by σ, where $\lambda \neq \alpha\beta$.

Proof. The first statement follows from remark 1.2. In (a) and (b), we seek a basis for I_σ.

(a) Conditions (1), (2) and (3) defining σ show that, with respect to the basis $\{e_i\}$, we have

$$\sigma|_Q = \begin{pmatrix} \alpha\lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \sigma|_L = \begin{pmatrix} \beta & 0 \\ 0 & 1 \end{pmatrix}$$

for some α, β, $\lambda \in k \setminus \{0\}$. Also, condition (4) of σ holds if and only if $\lambda \neq \alpha\beta$. Hence, σ is given by:

$$\sigma(x_1, x_2, x_3, x_4) = (\alpha\lambda x_1, \lambda x_2, \alpha x_3, x_4) \qquad \text{on } Q,$$
$$\sigma(0, x_2, x_3, 0) = (0, \beta x_2, x_3, 0) \qquad \text{on } L.$$

Therefore, $I_\sigma \supseteq \mathcal{R}$ where $\mathcal{R}$ is the linear span of the six linearly independent elements:

$$\alpha x_1 \otimes x_2 - x_2 \otimes x_1, \qquad x_4 \otimes x_2 - \lambda x_2 \otimes x_4,$$
$$\alpha x_3 \otimes x_4 - x_4 \otimes x_3, \qquad x_4 \otimes x_1 - \alpha\lambda x_1 \otimes x_4,$$
$$\lambda x_1 \otimes x_3 - x_3 \otimes x_1, \qquad x_3 \otimes x_2 - \beta x_2 \otimes x_3 + (\lambda - \alpha\beta)x_1 \otimes x_4.$$

We will show that $I_\sigma = \mathcal{R}$. Let $f \in I_\sigma \setminus \mathcal{R}$. Without loss of generality $f = \sum_{i=1}^{10} \alpha_i v_i$ is a linear combination of the elements:

$$v_1 = x_1 \otimes x_1, \quad v_2 = x_2 \otimes x_2, \quad v_3 = x_3 \otimes x_3, \quad v_4 = x_4 \otimes x_4, \quad v_5 = x_1 \otimes x_2,$$
$$v_6 = x_1 \otimes x_3, \quad v_7 = x_2 \otimes x_4, \quad v_8 = x_3 \otimes x_4, \quad v_9 = x_2 \otimes x_3, \quad v_{10} = x_1 \otimes x_4.$$

The vanishing of f at $(p,\sigma(p)) \in \Gamma_\sigma(Q)$, for all points p belonging to the four special lines on Q that meet L shows that $f = \alpha_9 v_9 + \alpha_{10} v_{10}$. Then evaluating f on $\Gamma_\sigma(L)$, and next at $(p,\sigma(p)) \in \Gamma_\sigma(Q)$ for a point p not lying on the four special lines, shows that $f = 0$. Thus, $I_\sigma = \mathcal{R}$.

(b) With respect to the basis $\{e_i\}$, we have

$$\sigma|_Q = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ \alpha\lambda & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \sigma|_L = \begin{pmatrix} 1 & 0 \\ 0 & \beta \end{pmatrix},$$

for some $\alpha,\ \beta,\ \lambda \in k \setminus \{0\}$. Again, condition (4) of σ holds if and only if $\lambda \neq \alpha\beta$. Hence, the map σ is given by:

$$\begin{aligned} \sigma(x_1,x_2,x_3,x_4) &= (x_4, \alpha x_2, \lambda x_3, \alpha\lambda x_1) \qquad &&\text{on } Q, \\ \sigma(0,x_2,x_3,0) &= (0, x_2, \beta x_3, 0) &&\text{on } L, \end{aligned}$$

and in this case we have $I_\sigma \supseteq \mathcal{R}$ where $\mathcal{R}$ is the linear span of the six linearly independent elements:

$$\begin{aligned} &\alpha x_1 \otimes x_3 - x_3 \otimes x_4, && x_4 \otimes x_3 - \lambda x_3 \otimes x_1, \\ &\alpha x_2 \otimes x_1 - x_4 \otimes x_2, && x_4 \otimes x_4 - \alpha\lambda x_1 \otimes x_1, \\ &\lambda x_1 \otimes x_2 - x_2 \otimes x_4, && x_2 \otimes x_3 - \beta x_3 \otimes x_2 - (\alpha\beta - \lambda) x_1 \otimes x_1. \end{aligned}$$

Then, as in (a), evaluating any $f \in I_\sigma \setminus \mathcal{R}$ at suitable points of Γ_σ proves that such an f must be zero. ∎

The algebras A_σ are defined in terms of geometric data. The next result shows that, conversely, the defining relations of the algebras determine the geometric data.

Lemma 1.4. $\Gamma_\sigma = \mathcal{V}(I_\sigma)$.

Proof. Since, by definition, $\Gamma_\sigma \subseteq \mathcal{V}(I_\sigma)$ we need only show the reverse inclusion. Let $p = (p_1,p_2,p_3,p_4)$ and $q = (q_1,q_2,q_3,q_4) \in \mathbb{P}(V^*)$. By 1.3 there are two cases to analyse.

(a) Suppose A_σ is given by the relations in 1.3 (a). The basis for I_σ found in proving 1.3(a) shows that $(p,q) \in \mathcal{V}(I_\sigma)$ if and only if

$$\left.\begin{pmatrix} x_4 & 0 & 0 & -\alpha\lambda x_1 \\ 0 & x_4 & 0 & -\lambda x_2 \\ 0 & 0 & x_4 & -\alpha x_3 \\ x_2 & -\alpha x_1 & 0 & 0 \\ x_3 & 0 & -\lambda x_1 & 0 \\ 0 & x_3 & -\beta x_2 & (\lambda - \alpha\beta) x_1 \end{pmatrix}\right|_p \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} = 0$$

(where the subscript $|_p$ beside the matrix means evaluation at p). Let M denote the matrix.

Suppose $(p,q) \in \mathcal{V}(I_\sigma)$. Then $\operatorname{rank}(M|_p) < 4$. We first show that $p \in Q \cup L$. The minor formed from the first three rows and the last row of M is $(\lambda - \alpha\beta) x_4^2 (x_1 x_4 + x_2 x_3)$, which must vanish at p. Therefore, if $x_4(p) \neq 0$ then $p \in Q$. On the other hand, if $x_4(p) = 0$ then the 4×4 minor:

$$\begin{vmatrix} 0 & 0 & 0 & 1 \\ x_2 & -\alpha x_1 & 0 & 0 \\ x_3 & 0 & -\lambda x_1 & 0 \\ 0 & x_3 & -\beta x_2 & (\lambda - \alpha\beta) x_1 \end{vmatrix} \quad = \quad (\alpha\beta - \lambda) x_1 x_2 x_3$$

also vanishes at p. So $p \in \mathcal{V}(x_2, x_4) \cup \mathcal{V}(x_3, x_4) \cup \mathcal{V}(x_1, x_4)$, which is a subset of $Q \cup L$.

We now claim that $q = \sigma(p)$. If not, then there are two independent solutions, q and $\sigma(p)$, to the equation, implying that $\operatorname{rank}(M|_p) < 3$. But this is false, since for each $p \in Q \cup L$ one may check that there is at least one 3×3 minor of $M|_p$ that is nonzero. It follows that $(p,\, q) \in \Gamma_\sigma$, from which we conclude $\mathcal{V}(I_\sigma) = \Gamma_\sigma$.

(b) Now suppose A_σ is given by the relations in 1.3 (b). One can check that application of the above argument to the matrix

$$\begin{pmatrix} \alpha\lambda x_1 & 0 & 0 & -x_4 \\ 0 & \lambda x_1 & 0 & -x_2 \\ 0 & 0 & \alpha x_1 & -x_3 \\ -\lambda x_3 & 0 & x_4 & 0 \\ -\alpha x_2 & x_4 & 0 & 0 \\ (\alpha\beta - \lambda)x_1 & \beta x_3 & -x_2 & 0 \end{pmatrix}$$

yields the desired result. ■

Remarks .

(a) The main reason that 1.4 holds is because of condition (4) that we imposed upon σ; i.e., the fact that σ is not the restriction of a linear map on $\mathbb{P}^3$.

(b) Notice that A_σ is never commutative. One explanation is as follows. If A_σ is commutative then case 1.3(a) occurs, forcing $\alpha = \lambda = \beta = 1$. But then σ is the restriction of the identity map on $\mathbb{P}^3$, contradicting condition (4) used in defining σ.

Example 1.5. The coordinate ring of quantum 2×2 matrices is denoted $\mathcal{O}_{\rm II}(\mathcal{M}_{\in}(\mathbb{C}))$, where $q \in \mathbb{C} \setminus \{0\}$ and $q^2 \neq 1$ ([FRT]). It is (isomorphic to) the algebra $\mathbb{C}[x_1, x_2, x_3, x_4]$ with the six defining relations:

$$\begin{aligned} x_2x_1 &= qx_1x_2, & x_3x_1 &= q^{-1}x_1x_3, & x_4x_1 &= x_1x_4, \\ x_4x_3 &= qx_3x_4, & x_4x_2 &= q^{-1}x_2x_4, & x_3x_2 - x_2x_3 &= (q - q^{-1})x_1x_4. \end{aligned}$$

By 1.3 and 1.4, $\mathcal{O}_{\rm II}(\mathcal{M}_{\in}(\mathbb{C})) \cong \mathcal{A}_\sigma$ where

$$\begin{aligned} \sigma(x_1, x_2, x_3, x_4) &= (x_1, q^{-1}x_2, qx_3, x_4) && \text{on } Q, \\ \sigma(0, x_2, x_3, 0) &= (0, x_2, x_3, 0) && \text{on } L. \end{aligned}$$

In particular, σ preserves the rulings on Q; moreover, σ is of finite order if and only if q is a root of unity.

Example 1.6. Families of algebras on n^2 generators containing $\mathcal{O}_{\rm II}(\mathcal{M}_{\setminus}(\mathbb{C}))$ are defined in [AST], [Su, §4b] and [T, §2]. The second family contains the third and, when $n = 2$, the third contains the first. Moreover, when $n = 2$, these families intersect with the subfamily $\{A_\sigma \colon \sigma$ preserves the rulings on Q and $\sigma|_L$ is the identity on $L\}$. However, our insistence on σ not being the restriction of a linear map on $\mathbb{P}^3$ excludes some of the algebras in [AST], [Su] and [T] ($n = 2$) from this subfamily – those omitted allow (in our notation) the possibility $\lambda - \alpha\beta = 0$.

Example 1.7. The coordinate ring of quantum symplectic 4-dimensional space is denoted $\mathcal{O}_{\rm II}(\mathfrak{sp}\mathbb{C}^4)$, where $q \in \mathbb{C}\setminus\{0\}$ and $q^2 \neq 1$ ([FRT]). It is the algebra $\mathbb{C}[x_1, x_2, x_3, x_4]$ with the six defining relations:

$$\begin{aligned} x_2x_1 &= q^{-1}x_1x_2, & x_3x_1 &= q^{-1}x_1x_3, & x_4x_1 &= q^{-2}x_1x_4, \\ x_4x_3 &= q^{-1}x_3x_4, & x_4x_2 &= q^{-1}x_2x_4, & x_3x_2 - q^{-2}x_2x_3 &= q^{-2}(q^{-1} - q)x_1x_4. \end{aligned}$$

By 1.3 and 1.4, $\mathcal{O}_{\mathrm{II}}(\mathfrak{sp}\mathbb{C}^4) \cong \mathfrak{A}_\sigma$ where

$$\begin{aligned} \sigma(x_1, x_2, x_3, x_4) &= (x_1, qx_2, qx_3, q^2x_4) \qquad &\text{on } Q, \\ \sigma(0, x_2, x_3, 0) &= (0, x_2, q^2x_3, 0) \qquad &\text{on } L. \end{aligned}$$

Here, σ preserves the rulings on Q; and σ is of finite order if and only if q is a root of unity.

Lemma 1.8. *The algebras A_σ are iterated Ore extensions, and hence skew-polynomial rings; that is, they are of the form $k[X_1; \mu_1, \delta_1] \cdots [X_4; \mu_4, \delta_4]$ where the μ_i are automorphisms and the δ_i are μ_i-derivations at each stage of the extension.*

Proof. For the algebras in 1.3(a) the result is easy to check by adjoining the variables in the order: x_1, x_2, x_4, x_3. For those in 1.3(b), setting

$$X_1 = \sqrt{\tfrac{\lambda-\alpha\beta}{4\alpha\beta\lambda}}\,(\sqrt{\alpha\lambda}x_1 - x_4),\ X_4 = \sqrt{\tfrac{\lambda-\alpha\beta}{4\alpha\beta\lambda}}\,(\sqrt{\alpha\lambda}x_1 + x_4),\ X_2 = x_2,\ X_3 = x_3$$

shows that each algebra of 1.3(b) is isomorphic to an algebra $k[X_1, X_2, X_3, X_4]$ with defining relations:

$$\begin{aligned} X_2X_4 &= \gamma X_4X_2, \qquad & X_2X_1 &= -\gamma X_1X_2, \qquad & X_4X_1 &= -X_1X_4, \\ X_4X_3 &= \gamma X_3X_4, \qquad & X_1X_3 &= -\gamma X_3X_1, \qquad & X_3X_2 - \nu X_2X_3 &= (X_1 + X_4)^2, \end{aligned}$$

where $\gamma = \sqrt{\frac{\lambda}{\alpha}}$ and $\nu = \frac{1}{\beta}$. By adjoining the variables in the order: X_1, X_2, X_4, X_3, the result is straightforward to verify. ■

Remarks .

(a) The basis $\{X_i\}$ in the proof of 1.8 is dual to the basis $\{e_1 + \sqrt{\alpha\lambda}e_4,\ e_2,\ e_3,\ e_1 - \sqrt{\alpha\lambda}e_4\}$. The latter set consists of eigenvectors for the linear map $\tau \in \mathrm{Aut}(\mathbb{P}^3)$ which satisfies $\tau|_Q = \sigma|_Q$.

(b) The algebras A_σ of 1.3(b) seem to depend on three parameters whereas the algebras $k[X_1, X_2, X_3, X_4]$, in the proof of 1.8, depend on only two. This can be explained as follows. Suppose σ_1, $\sigma_2 \in \mathrm{Aut}(Q \cup L)$ satisfy conditions (1) to (4), used in defining σ, with $\sigma_1|_L = \sigma_2|_L$, and that τ_1, $\tau_2 \in \mathrm{Aut}(\mathbb{P}^3)$ are linear maps with $\tau_i|_Q = \sigma_i|_Q$. Then $A_{\sigma_1} \cong A_{\sigma_2}$ if and only if there exists a linear map $\phi \in \mathrm{Aut}(\mathbb{P}^3)$ such that $\phi(Q) = Q$, $\phi(L) = L$ and $\tau_2 = \phi\tau_1\phi^{-1}$. (Loosely speaking, the algebras are determined up to conjugacy of σ.)

Corollary 1.9. *The algebra A_σ is a noetherian domain of global homological dimension* $\mathrm{gldim}(A_\sigma) = 4$, *with Hilbert series $H_{A_\sigma}(t) = (1-t)^{-4}$. Moreover, the Koszul complex is exact.*

Proof. The first sentence is proved using [MR, §1.2, 7.5.3, 7.9.16]. There are a number of ways to show exactness of the Koszul complex – one is to apply [LSV, 2.6], another is to exploit the PBW basis and apply [P]. ■

Notation 1.10.

(a) We will use the symbol τ to denote the linear automorphism of $\mathbb{P}^3$ such that $\tau|_Q = \sigma|_Q$. Notice that $\tau(L) = L$ since σ fixes both e_2 and e_3.

(b) If $u \in V$ and $\theta \in \mathrm{Aut}(\mathbb{P}^3)$ is a linear map, let u^θ denote the linear form $u \circ \theta$.

(c) For 1.3(a), let $\omega_1 = x_1$, $\omega_2 = x_4$;
for 1.3(b), let $\omega_1 = \sqrt{\alpha\lambda}x_1 - x_4$, $\omega_2 = \sqrt{\alpha\lambda}x_1 + x_4$.

(d) Let $\widetilde{\Omega} = x_3^\tau \otimes x_2 - x_2^\tau \otimes x_3$. The image of $\widetilde{\Omega}$ in A_σ will be denoted Ω.

Remark . If $k = \mathbb{C}$ and σ preserves the rulings on Q, then Ω is central if and only if $A_\sigma \cong \mathcal{O}_{\rm II}(\mathcal{M}_\in(\mathbb{C}))$ (see example 1.5). In this case, Ω is (a scalar multiple of) the quantum determinant.

Lemma 1.11.
(a) $L = \mathcal{V}(\omega_1, \omega_2)$.
(b) Ω *is the unique element in* A_2 *(up to a scalar multiple) that vanishes on* $\Gamma_\sigma(Q)$ *but not identically on* $\Gamma_\sigma(L)$*. Moreover,* $\mathcal{V}(\widetilde{\Omega}) \cap \Gamma_\sigma = \Gamma_\sigma(Q)$.
(c) *If* $u, v \in V$ *then* $u^\tau v - v^\tau u$ *is a scalar (possibly zero) multiple of* Ω.
(d) *If* $u, v \in V$ *and* $L \subset \mathcal{V}(uv)$ *then* $u^\tau v = v^\tau u$ *in* A_σ.
(e) ω_1*,* ω_2 *and* Ω *are normal in* A_σ.

Note . If $v \in V$ is an eigenvector for τ (i.e., $v^\tau \in kv$) and $v(L) = 0$ then 1.11(d) shows that $u^\tau v \in kvu$ for all $u \in V$. Hence, v is normal in A_σ. Furthermore, the first five relations in both 1.3(a) and 1.3(b) are consequences of 1.11(d).

Proof. (a) follows from the definitions.

(b) By definition of τ, we have $\widetilde{\Omega}(\Gamma_\sigma(Q)) = 0$. With notation as in 1.3, recall that $\lambda - \alpha\beta \neq 0$ due to σ not being the restriction of a linear map on $\mathbb{P}^3$. Taking $r = (0,1,1,0) \in L$ we find that $\widetilde{\Omega}(r, \sigma(r))$ is a nonzero scalar multiple of $\lambda - \alpha\beta$, and hence $\widetilde{\Omega}$ does not vanish identically on $\Gamma_\sigma(L)$. Now $\widetilde{\Omega}(p, \sigma(p)) = (x_3^\tau x_2^\nu - x_2^\tau x_3^\nu)(p)$ for all $p \in L$, where $\nu \in \mathrm{Aut}(\mathbb{P}^3)$ is a linear map such that $\nu|_L = \sigma|_L$. The polynomial $x_3^\tau x_2^\nu - x_2^\tau x_3^\nu$ is nonzero of degree 2, so, by Bézout's theorem, it can vanish at no more than two points of L. These points are $\{e_2, e_3\} = Q \cap L$, which proves $\mathcal{V}(\widetilde{\Omega}) \cap \Gamma_\sigma = \Gamma_\sigma(Q)$.

Furthermore, by the proof of 1.3 any element in $V \otimes V/I_\sigma$ that vanishes on $\Gamma_\sigma(Q)$ must belong to $kx_1 \otimes x_4 \oplus kx_2 \otimes x_3$ for 1.3(a) or $kx_1 \otimes x_1 \oplus kx_2 \otimes x_3$ for 1.3(b). Since $(x_2 \otimes x_3)(\Gamma_\sigma(Q)) \neq 0$ there can only be one such element (up to a scalar multiple) that does not vanish identically on $\Gamma_\sigma(L)$.

(c) follows from (b) since, by definition of τ, we have $u^\tau \otimes v - v^\tau \otimes u$ vanishes on $\Gamma_\sigma(Q)$; and (d) holds since the extra hypothesis ensures that now $u^\tau \otimes v - v^\tau \otimes u$ vanishes on $\Gamma_\sigma(L)$ also.

(e) Normality of ω_i follows from the previous results. (It can also be deduced from (d) since $\omega_i^\tau \in k\omega_i$ – see previous note.) A calculation shows that $\Omega x_i \in kx_i\Omega$, $1 \leq i \leq 4$. ■

2. Classification of Point and Line Modules

We observe in §2.1 that the point modules are in bijection with the points on the variety $Q \cup L$; in §2.2 we show that the line modules are in bijection with those lines in $\mathbb{P}^3$ that either lie on Q or meet L. It is shown that both point (§2.3) and line (§2.1) modules are quotients of A_σ by a left ideal generated by an appropriate subspace of $(A_\sigma)_1$.

2.1. **Preliminaries.** Henceforth, we assume σ is fixed and write A for A_σ. We define a grading on A by declaring the generators $x_1, \ldots, x_4$ to be of degree one; that is, A inherits the usual grading from $T(V)$. We will denote the homogeneous degree n part of A by A_n and identify V with its image, A_1, in A.

The Hilbert series of a graded module M will be denoted $H_M(t)$.

A *shift* of a $\mathbb{Z}$-graded module $M = \bigoplus_n M_n$ is a $\mathbb{Z}$-graded module $M[p]$ where $M[p]_n = M_{p+n}$ for some $p \in \mathbb{Z}$ and for all $n \in \mathbb{Z}$. Clearly, $H_{M[p]}(t) = t^{-p} H_M(t)$.

Definition . [ATV1, 2] Let M be a $\mathbb{Z}$-graded cyclic module over A. We say M is a *point*, resp. *line*, resp. *plane*, module if $H_M(t) = (1-t)^{-1}$, resp. $(1-t)^{-2}$, resp. $(1-t)^{-3}$.

Let u be a nonzero element in A_1. Then, since A is a domain, we have

$$H_{A/Au}(t) = H_A(t) - H_{Au}(t) = (1-t)^{-3}.$$

Thus, A/Au is a plane module. Conversely, suppose $M = \bigoplus_n M_n$ is a plane module. Since M is cyclic and $\dim(M_1) = 3$, it follows that there is a surjective homomorphism $A/Au \twoheadrightarrow M$ for some nonzero element $u \in A_1$. But A/Au is a plane module, whence $M \cong A/Au$. In particular, the plane modules are in bijection with the planes $\mathcal{V}(u)$ in $\mathbb{P}^3$.

Now, suppose K is a line module and M is a point module. A similar dimension argument shows there exist surjective homomorphisms:

$$\frac{A}{Au + Av} \twoheadrightarrow K \qquad \text{and} \qquad \frac{A}{Au' + Av' + Aw} \twoheadrightarrow M$$

for some nonzero elements u, v, u', v', $w \in A_1$. In this section we will show these maps are isomorphisms. However, for now, we can at least conclude that K defines a line, $\mathcal{V}(u, v)$, in $\mathbb{P}^3$ and M a point, $\mathcal{V}(u', v', w)$, in $\mathbb{P}^3$. In fact, 2.1 shows that more is true.

Proposition 2.1. *The point modules for A are in bijection with the points of the variety $Q \cup L$.*

Proof. Let $\pi_i \colon \mathbb{P}^3 \times \mathbb{P}^3 \to \mathbb{P}^3$ denote the projection map onto the i'th coordinate for $i = 1, 2$. Since $\mathcal{V}(I_\sigma) = \Gamma_\sigma$ is the graph of σ, we have $\pi_i|_{\mathcal{V}(I_\sigma)}$ is injective for $i = 1, 2$, and $\pi_1(\mathcal{V}(I_\sigma))$ equals $\pi_2(\mathcal{V}(I_\sigma))$ – both equal $Q \cup L$. The result now follows from [ATV1, §3]. ■

Remark 2.2. By [ATV1, §3], the action of the algebra A on a point module $M(p)$, corresponding to a point $p \in Q \cup L$, can be described as follows: if $x \in A_1$ and v_i is a basis for $M(p)_i$ for $i \geq 0$ then $x \cdot v_i = x(\sigma^{-i}(p)) v_{i+1}$ for all $i \geq 0$.

It is demonstrated in [LS] that, in some cases, it is possible for information on point, line and plane modules to be extracted from certain homological data concerning the algebra. Following this approach, we employ terminology from [LS, §1]. Corollary 1.9 and proposition 2.3 show that A satisfies a sufficient number of the hypotheses on the algebras discussed in [LS, §2] to enable the line modules to be described in a natural way (theorem 2.4).

Proposition 2.3. *The algebra A is Auslander-regular of dimension 4 and satisfies the Cohen-Macaulay property.*

Proof. The elements ω_1, ω_2, x_2, x_3 form a regular normalizing sequence in A. The result then follows by applying [L, §5.10] and corollary 1.9. ■

Theorem 2.4. [LS, §2] *Let A be a noetherian, graded k-algebra, generated in degree 1, with Hilbert series $H_A(t) = (1-t)^{-4}$, which is Auslander-regular of dimension 4 and satisfies the Cohen-Macaulay property. Then the line modules are in bijection with those lines ℓ in $\mathbb{P}^3$ such that $\ell = \mathcal{V}(u, v)$ where u, $v \in A_1$ are linearly independent elements satisfying $A_1 u \cap A_1 v \neq 0$. The line module corresponding to such a line ℓ is (isomorphic to) $A/Au + Av$.* ■

Notation . The symbols $M(p)$ and $M(\ell)$ will denote the point module, resp. line module, corresponding to the point p or line ℓ.

2.2. **Geometric Classification of Line Modules.**

Theorem 2.5. *There is a bijection between the set of isomorphism classes of line modules over A and the collection of lines in $\mathbb{P}^3$ that either lie on Q or meet L.*

We will prove the theorem in two parts, making use of 2.4, but first we need a lemma.

Lemma 2.6. *Let $\ell := \mathcal{V}(u, v)$ be a line in $\mathbb{P}^3$, where $u, v \in V$. Suppose $\ell \cap L = \emptyset$ and $\ell \not\subset Q$. Then there exists a nonzero element $w \in ku \oplus kv$ such that $\mathcal{V}(w) \cap Q$ is a nondegenerate conic in the plane $\mathcal{V}(w)$ and $\mathcal{V}(w) \cap L \not\subset Q$. Furthermore, there are many such w.*

Proof. Let $\check{\mathbb{P}}^3$ be the dual space consisting of the planes in $\mathbb{P}^3$, and let $\check{\ell} \subset \check{\mathbb{P}}^3$ be the line consisting of the planes containing ℓ. Notice that $L \not\subset H$ for all $H \in \check{\ell}$ (otherwise $L \cap \ell \neq \emptyset$).

Since $\ell \not\subset Q$ we have $\ell \cap Q = \{x, y\}$, where possibly $x = y$. Let ℓ_1, ℓ_2 be the two lines on Q which pass through x. Consider a conic $H \cap Q$ for any $H \in \check{\ell}$. If the conic is degenerate then it contains a line passing through $x \in H \cap Q$, and so contains either ℓ_1 or ℓ_2. Thus, if there were three distinct $H \in \check{\ell}$ such that $H \cap Q$ is degenerate, then two of them, say H and H', must contain a common ℓ_i ($i = 1$ or 2). But then, $\ell_i \subset H \cap H' = \ell$, and so $\ell \subset Q$ which is false. Therefore, the set $\{H \in \check{\ell} \colon H \cap Q \text{ is nondegenerate}\}$ is a dense open subset of $\check{\ell}$.

Now consider the subset $Y := \{H \in \check{\ell} \colon H \cap L \subset Q\}$ of $\check{\ell}$. If $|Y| \geq 3$ then, since $|Q \cap L| = 2$, there would exist $H, H' \in Y$, $H \neq H'$, with $H \cap L = H' \cap L$. It would then follow that $\ell \cap L = (H \cap H') \cap L = H \cap L \neq \emptyset$, which is a contradiction. Thus, $|Y| \leq 2$ implying that $\{H \in \check{\ell} \colon H \cap L \not\subset Q\}$ is a dense open subset of $\check{\ell}$.

Hence, the set $\{H \in \check{\ell} \colon H \cap Q \text{ is a nondegenerate conic and } H \cap L \not\subset Q\}$ is dense in $\check{\ell}$, which completes the proof. ■

Proposition 2.7. *Let $A/Au + Av$ be a line module where $u, v \in A_1$. Then the line $\ell := \mathcal{V}(u, v)$ either lies on the quadric Q or meets the line L.*

Proof. Comparing the Hilbert series of A with that of a line module, we see that $\dim(A_1 u + A_1 v) = 7$, so therefore $A_1 u \cap A_1 v \neq 0$. (In fact, this is a minor part of 2.4, but is true for these elementary dimension reasons.) In particular, there exist nonzero elements $a, b \in A_1$ such that $av = bu$, or equivalently

$$(a \otimes v - b \otimes u)(\Gamma_\sigma) = 0. \qquad (*)$$

Suppose for a contradiction that $\ell \cap L = \emptyset$ and $\ell \not\subset Q$.

By 2.6 we can pick a plane $H = \mathcal{V}(w)$ with $0 \neq w \in ku \oplus kv$, such that $H \cap Q$ is a nondegenerate conic in H and $H \cap L = \{p\}$ for some $p \notin Q$. Moreover, we can assume $w = u$, since if w, $w' \in ku \oplus kv$ are nonzero and linearly independent then $A_1 w \cap A_1 w' \neq 0$ also.

Now, $\{p\}$ and the nondegenerate conic $H \cap Q$ both lie in the plane H. Since $p \notin H \cap Q$ there are exactly two lines in H passing through p which are tangent to $H \cap Q$. It follows that there are many choices of distinct points $q, r, s \in H \cap Q$ such that p, q, r are collinear but with p, q, r, s not collinear. We choose such distinct points $q, r, s \in H \cap Q$ such that $\{q, r, s\} \cap \ell \cap Q$ is empty.

By construction, u vanishes at p, q, r, s whereas v is nonzero at these four points. So, applying $(*)$ to $(\sigma^{-1}(p), p), \ldots, (\sigma^{-1}(s), s)$ in turn implies that a vanishes at $\sigma^{-1}(p)$, $\sigma^{-1}(q)$, $\sigma^{-1}(r)$, $\sigma^{-1}(s)$. In particular, these four points are coplanar. We will prove that this is, in fact, not the case.

Since the map τ (defined in 1.10) is linear, it follows that $\tau\sigma^{-1}(p)$, q, r, s are coplanar. But q, r, s span H, and $\tau(L) = L$, so $\tau\sigma^{-1}(p) \in H \cap L = \{p\}$. Therefore, $\sigma^{-1}(p) = \tau^{-1}(p)$ so that the linear maps $\sigma^{-1}|_L$ and $\tau^{-1}|_L$ agree at three distinct points, e_2, e_3, p, of L. It follows that $\sigma^{-1}|_L = \tau^{-1}|_L$, and hence $\sigma = \tau|_{Q \cup L}$, contradicting condition (4) used in defining σ. ■

Proposition 2.8. *Let ℓ be a line in $\mathbb{P}^3$ that either lies on Q or meets the line L. Then there exist elements $u, v \in A_1$ such that $\ell = \mathcal{V}(u, v)$ and $A/Au + Av$ is a line module.*

Proof. We will show there exist nonzero elements u, v, a, $b \in A_1$ such that $\ell = \mathcal{V}(u, v)$ and $av = bu$, and invoke 2.4 to finish the proof.

Firstly, if $\ell \cap L \neq \emptyset$, then there exist $u, v \in V$ such that $\ell = \mathcal{V}(u, v)$ and $u(L) = 0$. By 1.11(d), it follows that $u^\tau v = v^\tau u$.

Now suppose $\ell \cap L = \emptyset$ and $\ell \subset Q$. Then the distinguished line L does not lie in any plane that contains ℓ.

Choose two distinct lines ℓ_1, ℓ_2 on Q, in the ruling on Q which does not contain ℓ, such that $\ell_i \cap L = \emptyset$, for $i = 1, 2$. We take $u \in V$ to be the plane determined by the lines ℓ and ℓ_1, and similarly $v \in V$ to be that determined by ℓ and ℓ_2. Then $\ell = \mathcal{V}(u, v)$, and $u(L) \neq 0$ and $v(L) \neq 0$.

There is exactly one point $r \in L$ such that $u(\sigma(r)) = 0$ and, since $\mathcal{V}(u) \cap Q = \ell \cup \ell_1 \subset Q \setminus L$, we have $r \notin Q$. Put $\{q\} = \mathcal{V}(v) \cap L$. By hypothesis, $q \neq \sigma(r)$.

Now u/v is a rational function on both Q and L with associated divisors

$$\operatorname{div}_Q\left(\frac{u}{v}\right) = \ell_1 - \ell_2 \qquad \text{and} \qquad \operatorname{div}_L\left(\frac{u}{v}\right) = \sigma(r) - q.$$

Define a linear form $a \in V$ to be such that $r \in \mathcal{V}(a)$ and $\sigma^{-1}(\ell_1) \subset \mathcal{V}(a)$. Since $\sigma^{-1}(\ell_1) \subset Q \cap \mathcal{V}(a)$, which is a conic in $\mathcal{V}(a)$, it follows that the divisor of zeros of a on Q is $(a)_0 = \ell_3 + \sigma^{-1}(\ell_1)$ for some line $\ell_3 \subset Q$. Next, define a linear form $b \in V$ by requiring that the divisor of zeros of b on Q is $(b)_0 = \ell_3 + \sigma^{-1}(\ell_2)$.

The rational function a/b on Q has divisor

$$\operatorname{div}_Q\left(\frac{a}{b}\right) = \sigma^{-1}(\ell_1) - \sigma^{-1}(\ell_2).$$

However,

$$\operatorname{div}_Q\left(\frac{u}{v}\right)^\tau = \sigma^{-1}\left(\operatorname{div}_Q\left(\frac{u}{v}\right)\right) = \sigma^{-1}(\ell_1) - \sigma^{-1}(\ell_2).$$

Hence, replacing a by a suitable scalar multiple we have $a/b = (u/v)^\tau$ on Q; that is,

$$(av^\tau - bu^\tau)(p) = 0 \qquad \text{for all } \ p \in Q.$$

More precisely, we have

$$(a \otimes v - b \otimes u)\,(p, \sigma(p)) = 0 \qquad \text{for all } \ p \in Q.$$

It remains to show this also holds on L. Since it is already true at $p = e_i$ for $i = 2, 3$, by Bézout's theorem we need only show it holds at some $p \in L \setminus \{e_2, e_3\}$. But, by choice of $r \in L \setminus Q$ above, we have $u(\sigma(r)) = 0$ and $a(r) = 0$, implying that $(a \otimes v - b \otimes u)(r, \sigma(r)) = 0$. It follows that $av = bu$ in A. ■

Note . Suppose $u, v \in V$ satisfy the hypotheses of 2.7. Then the point $(a, b) \in \mathbb{P}(V \times V)$ such that $av = bu$ (or equivalently, such that $(a \otimes v - b \otimes u)(\Gamma_\sigma) = 0$) is unique (see the proof of [LS, 4.3]).

2.3. **Point Modules.** By remark 2.2, if $p \in \mathcal{V}(u, v)$, where u, v are nonzero, linearly independent elements of V, then $M(p)$ is a quotient of $A/Au + Av$. It then follows from proposition 2.1 and theorem 2.5 that every point module is a quotient of a line module. Hence, if $\mathcal{V}(u, v) \equiv \ell$ corresponds to a line module then we have the short exact sequence

$$0 \longrightarrow K \longrightarrow M(\ell) \longrightarrow M(p) \longrightarrow 0$$

for some A-module $K \subset M(\ell)$. In fact, K is a shift of a line module as can be seen from the following result.

Proposition 2.9. *Let $p = \mathcal{V}(u, v, w) \in Q \cup L$ where u, v, w are nonzero, linearly independent elements of V. Then the corresponding point module $M(p)$ is given by $M(p) \cong A/Au + Av + Aw$.*

Proof. Since $\mathcal{V}(I_\sigma)$ is the graph of the automorphism σ, we can apply [LSV, 4.1.1]. ■

The line corresponding to the shifted line module K, in the above short exact sequence, is identified in terms of the point p and the line ℓ in §5.

Remarks .

(a) Given an automorphism of an algebra, a procedure for twisting the algebra, and the modules over it, by this automorphism, is given in [ATV2, §8]. The category of graded modules over the twisted algebra is equivalent to the category of graded modules over the original algebra ([ATV2, 8.5]). For both kinds of σ there exists such a twist of A_σ that yields the algebra $A_{\sigma \circ \tau^{-1}}$ which is of type 1.3(a) (where τ as defined in 1.10). Notice that $\sigma \circ \tau^{-1}|_Q$ is the identity map on Q. Here the twist map depends on ω_2 and, by lemma 1.11(d), it is given by τ. If $M(X) = M$ is a linear module over A_σ, where X is a point/line/plane, then $M^\tau[d]$ corresponds to X but $M[d]^\tau$ corresponds to $\tau^{-d}X$. In particular, we could work over $A_\sigma^\tau := A_{\sigma \circ \tau^{-1}}$ and "twist" the results back to A_σ. However, the work for A_σ^τ is *not* that much simpler, and, in fact, studying A_σ^τ alone is insufficient for proving many results on annihilators of such modules $M(X)$, since the order of $\sigma|_Q$ plays a substantial role in determining $\text{ann}M(X)$. For instance, over A_σ^τ, $M(p) \cong A_\sigma^\tau/\text{ann}M(p)$ for all $p \in Q$, and the latter (as an algebra) is isomorphic to a polynomial ring in one variable ([LS, §5]) – over arbitrary A_σ this is false. For these reasons, we consider arbitrary A_σ throughout the paper.

(b) The work in [LSV] applies to twists of A_σ, and "twisting" their results back to A_σ shows that a line ℓ through $p \in L \cup (Q \cap \mathcal{V}(\omega_2))$ corresponds to a line module if and only if ℓ is contained in $\mathcal{V}(\omega_2)$ or a certain quadric Q_p. Our results show that Q_p is independent of p and $Q_p = Q$.

3. The Algebra $A/A\Omega$

Recall that the normal element Ω (defined in 1.10) is the unique element of A_2 (up to a scalar multiple) that vanishes on $\Gamma_\sigma(Q)$ but not on $\Gamma_\sigma(L)$. In this section we show that $A/A\Omega$ is a twisted homogeneous coordinate ring ([AV]) of Q with respect to σ

and a suitable invertible sheaf $\mathcal{L}$ on Q. The general construction is taken from [ATV1] and will show that $A/A\Omega$ is a domain.

For convenience, in this section only, σ will denote $\sigma|_Q$.

Notation .

(a) Let $j\colon Q \hookrightarrow \mathbb{P}^3$ be the inclusion map and put $\mathcal{L} = |^*\mathcal{O}_{\mathbb{P}^\ni}(\infty) = \mathcal{O}_{\mathcal{Q}}(\infty)$.
(b) For each $n \in \mathbb{Z}$, write $\mathcal{L}^{\sigma^\backslash} = (\sigma^\backslash)^*\mathcal{L}$.
(c) Write $\mathcal{L}_\prime = \mathcal{O}_{\mathcal{Q}}$ and set $\mathcal{L}_\backslash = \mathcal{L} \otimes \cdots \otimes \mathcal{L}^{\sigma^{\backslash-\infty}}$ for $n \geq 1$.
(d) Write $B_n = H^0(Q, \mathcal{L}_\backslash)$ and define $B = \bigoplus_{n\geq 0} B_n$.

We give B the structure of a graded ring by defining the multiplication as follows: if $u \in B_n$, $v \in B_m$ and v^{σ^n} denotes the image of v (see remark 3.1) in $H^0(Q, \mathcal{L}_\Updownarrow^{\sigma^\backslash})$ then

$$u \cdot v = \mu_{n,m}(u \otimes v^{\sigma^n}) \quad \in B_{n+m}$$

where $\mu_{n,m}\colon H^0(Q, \mathcal{L}_\backslash) \otimes \mathcal{H}^\prime(\mathcal{Q}, \mathcal{L}_\Updownarrow^{\sigma^\backslash}) \to \mathcal{H}^\prime(\mathcal{Q}, \mathcal{L}_{\backslash+\Updownarrow})$ is the natural map.

We will show that $B \cong A/A\Omega$.

Remark 3.1. If $U \subset Q$ is open, then by definition of $\mathcal{L}^\sigma$, we have $\mathcal{L}^\sigma(\mathcal{U}) = \mathcal{L}(\sigma\mathcal{U})$. Thus, if $v \in \mathcal{L}^\sigma(\mathcal{U})$ then $v \circ \sigma \in \mathcal{L}(\mathcal{U})$. In the case that $U = Q$, then $\mathcal{L}^\sigma(\mathcal{Q}) = \mathcal{L}(\mathcal{Q})$ and so, for this reason, we take $v \in \mathcal{L}(\mathcal{Q})$ and $v \circ \sigma = v^\sigma \in \mathcal{L}^\sigma(\mathcal{Q})$. Moreover, if $v_i \in B_1$ then the element $v_1 \cdot v_2 \in B_2$ is the form on Q defined by $(v_1 \cdot v_2)(p) = v_1(p)v_2^\sigma(p) = v_1(p)v_2(\sigma(p))$ for all $p \in Q$.

Lemma 3.2. *There is a k-algebra homomorphism $\phi\colon A \to B$.*

Proof. Since Q is defined by one quadratic relation it follows that

$$B_1 = H^0(Q, \mathcal{L}) \cong \mathcal{H}^\prime(\mathbb{P}^\ni, \mathcal{O}_{\mathbb{P}^\ni}(\infty)) = \|\S_\infty \oplus \|\S_\in \oplus \|\S_\ni \oplus \|\S_\triangle.$$

So $A_1 \cong B_1$ via $\phi\colon x_i \mapsto x_i$. We claim ϕ induces a homomorphism from A to B. For if $f = \sum \lambda_{ij}x_ix_j = 0$ in A then $\phi(f) = \sum \lambda_{ij}x_i \cdot x_j = 0$ in B, because f vanishes on $\Gamma_\sigma(Q) \subset \Gamma_\sigma$, that is, $\phi(f)(p) = \sum \lambda_{ij}x_i(p)x_j(\sigma(p)) = 0$ for all $p \in Q$, by remark 3.1. ∎

Lemma 3.3. *B is generated by B_1 as a k-algebra.*

Proof. We need to prove that $B_{n+1} = B_nB_1$ for all $n \geq 1$. However, by the definition of multiplication in B, this is equivalent to showing that the cokernel of the multiplication map

$$\mu_{n,1}\colon H^0(Q, \mathcal{L}_\backslash) \otimes \mathcal{H}^\prime(\mathcal{Q}, \mathcal{L}^{\sigma^\backslash}) \to \mathcal{H}^\prime(\mathcal{Q}, \mathcal{L}_{\backslash+\infty})$$

is zero for all $n \geq 1$. But this is a consequence of [M, theorem 2] since $\mathcal{L}^\sigma \cong \mathcal{L}$, $\mathcal{L}_\backslash \cong \mathcal{O}_{\mathcal{Q}}(\backslash) \cong \mathcal{L}^\backslash$ and

$$H^i(Q, \mathcal{L}_\backslash \otimes (\mathcal{L}^{\sigma^\backslash})^{-\rangle}) = \mathcal{H}^\rangle(\mathcal{Q}, \mathcal{O}_{\mathcal{Q}}(\backslash - \rangle)) = \prime \quad \text{for all } \backslash, \rangle \geq \infty$$

(computed from the long exact sequence in cohomology applied to the short exact sequence $\quad 0 \to \mathcal{O}_{\mathbb{P}^\ni}(\backslash - \in) \to \mathcal{O}_{\mathbb{P}^\ni}(\backslash) \to \mathcal{O}_{\mathcal{Q}}(\backslash) \to \prime$). ∎

Corollary 3.4. *The map ϕ is surjective and $B \cong A/A\Omega$.*

Proof. It follows from 3.2 and 3.3 that $B = \phi(A)$. In addition, $\Omega(\Gamma_\sigma(Q)) = 0$ so $\Omega \in \ker(\phi)$. Thus there is a surjective homomorphism $A/A\Omega \twoheadrightarrow B$.

Now, since $\mathcal{L} \cong \mathcal{L}^\sigma$, we have $\dim H^0(Q, \mathcal{L}_\backslash) = \dim H^0(Q, \mathcal{L}^\backslash)$ and, moreover, the algebra $\bigoplus H^0(Q, \mathcal{L}^\backslash)$ is isomorphic to the homogeneous coordinate ring of Q (since the latter is integrally closed). Therefore B has the same Hilbert series as the homogeneous coordinate ring of Q which, in turn, has the same Hilbert series as $A/A\Omega$ (since A is a domain). Hence $B \cong A/A\Omega$ as desired. ■

Lemma 3.5. *The algebra $A/A\Omega$ is a domain.*

Proof. By 3.4 we need only show that B is a domain. Suppose there exist elements $u \in B_n$, $v \in B_m$ such that $0 = u \cdot v \in B_{n+m}$. By remark 3.1, this implies that $0 = uv^{\sigma^n} \in H^0(Q, \mathcal{L}^{\backslash+\Updownarrow})$. As in the proof of 3.4, $\bigoplus H^0(Q, \mathcal{L}^{\backslash})$ is isomorphic to the homogeneous coordinate ring of Q, which is a domain since Q is irreducible. Therefore $u = 0$ or $v^{\sigma^n} = 0$, from which the result follows. ■

Remark .
Consider τ (defined in 1.10(a)) as an automorphism of V via the rule: $v^\tau = v \circ \tau$ for $v \in V$ (see 1.10(b)). Let R denote the homogeneous coordinate ring of Q and extend τ to an automorphism of R in the obvious way. Define a new multiplication $*$ on R via: $a * b = ab^{\tau^m}$, where $a, b \in R$ are homogeneous elements of degrees m, n respectively. This multiplication is "the same" as that used to define B from $\bigoplus H^0(Q, \mathcal{L}^{\backslash}) = \mathcal{R}$ and the "twisted" algebra, R^τ, so obtained, is isomorphic to $A/A\Omega$. Consequently, the category of graded modules over $A/A\Omega$ ($\cong B$) is equivalent to the category of graded modules over R ([ATV2, 8.5]).

4. Annihilators of Point and Line Modules

The main result of this section shows that when σ has infinite orbit at "most" points of Q then $\mathrm{ann} M(\ell) = A\Omega$ where ℓ is any line on Q that does not meet L.

Remark 4.1. By [LS, §1, §2], point, line and plane modules are critical for GK-dimension, and hence have prime annihilators (adapt the argument of [St, 3.9] or alternatively use [ATV2, 2.30(vi)]).

For this reason, the annihilators of linear modules feature prominently in [SS] in their search for the primitive ideals of the Sklyanin algebra, and this suggests they warrant study here too.

Lemma 4.2. *Let $\omega_1, \omega_2 \in A_1$ be the normal elements defined in 1.10.*

(*a*) *The point module $M(p)$ is annihilated by ω_1 and ω_2 if and only if $p \in L$.*
(*b*) *The point module $M(p)$ is annihilated by Ω if and only if $p \in Q$.*

Proof. The result follows from 1.11, 2.1 and 2.2. ■

Note . By §2.2, $M(L) \cong A/A\omega_1 + A\omega_2$ which (as an algebra) is isomorphic to a quantum plane. Hence, the points on L can be viewed as corresponding to point modules over a quantum plane.

Remark 4.3. Let $M(\ell)$ be a line module and suppose there are infinitely many points on the line ℓ that correspond to point modules. Then it follows from [S, §1] that $\mathrm{ann} M(\ell) = \bigcap \mathrm{ann} M(p)$, where the intersection runs over infinitely many $p \in \ell$ that correspond to point modules.

Lemma 4.4.

(*a*) *The line module $M(\ell)$ is annihilated by ω_1 and ω_2 if and only if $\ell = L$.*
(*b*) *The line module $M(\ell)$ is annihilated by Ω if and only if $\ell \subset Q$.*

Proof. (a) follows from §2.2.

(b) If $\ell \subset Q$ then lemma 4.2 and remark 4.3 show that Ω annihilates $M(\ell)$.

Conversely, if $\ell \not\subset Q$ then, by 2.5, ℓ meets the line L. Therefore, we can assume $\ell = \mathcal{V}(u, v)$ where u, v are nonzero linearly independent elements of V such that $u(L) = 0$.

If $\Omega \in \text{ann}M(\ell)$ then there exist elements a, $b \in V$ such that

$$(a \otimes v - b \otimes u)(\Gamma_\sigma(Q)) = 0 \quad \text{whereas} \quad (a \otimes v - b \otimes u)(\Gamma_\sigma(L)) \neq 0.$$

In particular, the degree 2 polynomial $av^\tau - bu^\tau$ vanishes on Q (where τ as defined in 1.10). Since Q is irreducible, $av^\tau - bu^\tau \in k(x_1x_4 + x_2x_3)$; but $av^\tau - bu^\tau$ also vanishes on the line $\tau^{-1}(\ell)$ which is not contained in Q, so that $av^\tau - bu^\tau = 0$ in the polynomial ring. Thus, $u^\tau \in ka$ or kv^τ, and $v^\tau \in kb$ or ku^τ. Since u and v are linearly independent and τ is bijective we have $u^\tau \in ka$ and $v^\tau \in kb$. Therefore,

$$a \otimes v - b \otimes u = \lambda_1 u^\tau \otimes v - \lambda_2 v^\tau \otimes u$$

for some $\lambda_1, \lambda_2 \in k$. However, since $u^\tau(L) = u(L) = 0$, this expression vanishes on $\Gamma_\sigma(L)$, contradicting the choice of a and b. We conclude that $\Omega \notin \text{ann}M(\ell)$ as desired. ■

In [ATV2], a regular algebra of dimension 3 is shown to be a finite module over its centre if and only if the associated automorphism of the scheme E (see introduction) has finite order. This same dichotomy also occurs for the Sklyanin algebra ([LS], [S]) and therefore, as might be expected, to determine the annihilators more precisely, we need to make some assumption on the order of σ. However, to assume σ has infinite order at every point (other than $e_1, \ldots, e_4$) of $Q \cup L$, excludes $\mathcal{O}_{\text{II}}(\mathcal{M}_\in(\mathbb{C}))$ given in 1.5, for which $\sigma|_L$ is the identity on L. In examples 1.5 and 1.7, taking q not to be a root of unity is equivalent to insisting σ have infinite orbit at all points of $Q \setminus \{e_j\}_{j=1}^4$ only. In fact, this is the generic case: that is, in the notation of 1.3, when σ preserves the rulings on Q, then $|\langle\sigma\rangle \cdot p| = \infty$ for all $p \in Q \setminus \{e_j\}_{j=1}^4$ if and only if both α and λ are not roots of unity; when σ interchanges the rulings on Q, then $|\langle\sigma\rangle \cdot p| = \infty$ for all $p \in Q \setminus \{e_j\}_{j=1}^4$ if and only if α/λ is not a root of unity. Thus, 4.5 and 4.6 apply to both $\mathcal{O}_{\text{II}}(\mathcal{M}_\in(\mathbb{C}))$ and $\mathcal{O}_{\text{II}}(\mathfrak{sp}\mathbb{C}^4)$ when q is not a root of unity.

We recall from §1 the existence of exactly four special lines on Q that meet L.

Proposition 4.5. *Suppose* $|\langle\sigma\rangle \cdot p| = \infty$ *for all* $p \in Q \setminus \{e_j\}_{j=1}^4$ *and let* ℓ *be a line on* Q. *If* $\ell \cap L = \emptyset$, *then* $\text{ann}M(\ell) = A\Omega$.

Proof. Since $\Omega \in \text{ann}M(\ell) \subseteq \text{ann}M(p)$ for all $p \in \ell$, we have

$$\frac{A}{A\Omega} \twoheadrightarrow \frac{A}{\text{ann}M(\ell)} \twoheadrightarrow \frac{A}{\text{ann}M(p)} \qquad \text{for all } \; p \in \ell.$$

Suppose for a contradiction that $\text{ann}M(\ell) \neq A\Omega$. Then, since $A\Omega$ is prime (by §3) and $\text{GKdim}(A/A\Omega) = 3$, we have $\text{GKdim}(A/\text{ann}M(\ell)) = 2$. However, since the points on ℓ have infinite orbit, an argument similar to that of [LS, 5.11] shows that $\text{GKdim}(A/\text{ann}M(p)) \geq 2$ for all $p \in \ell$. But, by remark 4.1, $\text{ann}M(\ell)$ is prime, so

$$\text{ann}M(\ell) = \text{ann}M(p) \qquad \text{for all } \; p \in \ell. \qquad (*)$$

Now, ℓ meets one of the four special lines ℓ' on Q. By remark 1.2 there exists a plane $\mathcal{V}(w)$ such that (i) $\mathcal{V}(w) \cap Q =$ two special lines, and (ii) $\sigma^i(\ell') \subseteq \mathcal{V}(w) \cap Q$ for all i. Therefore $w \in \text{ann}M(q)$ for all $q \in \ell'$.

In particular, setting $r = \ell \cap \ell'$ and applying $(*)$ we have $w \in \operatorname{ann} M(r) = \operatorname{ann} M(\ell)$, which proves $w(\ell) = 0$. That is, $\ell \subset \mathcal{V}(w) \cap Q$ and so must be a special line on Q, contradicting our hypothesis on ℓ. ■

Proposition 4.6. *Suppose σ preserves the rulings on Q and that $|\langle\sigma\rangle \cdot p| = \infty$ for all $p \in Q \setminus \{e_j\}_{j=1}^4$. Let $\ell := \mathcal{V}(u, v)$, where $u, v \in V$, be one of the four special lines on Q. Then*

(*a*) $\operatorname{ann} M(\ell) = Au + Av$;
(*b*) $\operatorname{ann} M(e_i) = Ax_j + Ax_m + Ax_n$ *where* $\{i, j, m, n\} = \{1, 2, 3, 4\}$;
(*c*) *if* $p \in \ell \setminus \{e_j\}_{j=1}^4$ *then* $\operatorname{ann} M(p) = \operatorname{ann} M(\ell)$.

Proof. (a) By 2.4, $\operatorname{ann} M(\ell) \subseteq Au + Av$ so we need only show the reverse inclusion. Since $\sigma(\ell) = \ell$ we have $u(\sigma^i(\ell)) = 0 = v(\sigma^i(\ell))$ for all i, so applying remark 2.2 we obtain $Au + Av \subseteq \operatorname{ann} M(p)$ for all $p \in \ell$. It then follows from remark 4.3 that $Au + Av \subseteq \operatorname{ann} M(\ell)$.

(b) Since $\sigma(e_j) = e_j$ the result follows from remark 2.2.

(c) We have $\operatorname{ann} M(\ell) \subset \operatorname{ann} M(p)$, and also by (a), $M(\ell) \cong A/\operatorname{ann} M(\ell)$. Therefore

$$\operatorname{GKdim}\left(\frac{A}{\operatorname{ann} M(p)}\right) \leq \operatorname{GKdim}\left(\frac{A}{\operatorname{ann} M(\ell)}\right) = 2.$$

However, as in the proof of 4.5, since p has infinite orbit, $\operatorname{GKdim}(A/\operatorname{ann} M(p)) \geq 2$ – whence equals 2. Since $\operatorname{ann} M(\ell)$ is prime, it follows that $\operatorname{ann} M(p) = \operatorname{ann} M(\ell)$. ■

In reading 4.7, one should note that $\mathcal{V}(x_2) \cap \mathcal{V}(x_3) \not\subset Q$ (see §1).

Proposition 4.7. *Suppose σ interchanges the two rulings on Q and that $|\langle\sigma\rangle \cdot p| = \infty$ for all $p \in Q \setminus \{e_j\}_{j=1}^4$. Let ℓ be one of the four special lines on Q. Then*

(*a*) *for* $\ell \subset \mathcal{V}(x_i)$ *where* $i = 2, 3$ *we have* $\operatorname{ann} M(\ell) = \langle x_i \rangle$;
(*b*) $\operatorname{ann} M(e_1) = \operatorname{ann} M(e_4) = \langle x_2, x_3 \rangle$;
(*c*) *for* $i = 2, 3$ *we have* $\operatorname{ann} M(e_i) = Ax_j + Ax_m + Ax_n$ *where* i, j, m, n *are distinct*;
(*d*) *if* $p \in \ell \setminus \{e_j\}_{j=1}^4$ *then* $\operatorname{ann} M(p) = \operatorname{ann} M(\ell)$.

Proof. Since $\ell \subset \mathcal{V}(x_2) \cup \mathcal{V}(x_3)$, we assume $\ell \subset \mathcal{V}(x_3)$. Then $x_3 \in \operatorname{ann} M(p)$ for all $p \in \ell$ and so, by 4.3, $x_3 \in \operatorname{ann} M(\ell)$. If $p \in \ell$ has infinite orbit, then (as above) we have

$$2 = \operatorname{GKdim}\left(\frac{A}{\langle x_3 \rangle}\right) \geq \operatorname{GKdim}\left(\frac{A}{\operatorname{ann} M(\ell)}\right) \geq \operatorname{GKdim}\left(\frac{A}{\operatorname{ann} M(p)}\right) \geq 2.$$

Hence, since $\operatorname{ann} M(\ell)$ is prime, we have $\operatorname{ann} M(p) = \operatorname{ann} M(\ell)$, which proves (d). To prove (a) it suffices to show $\langle x_3 \rangle$ is prime – we leave this for now.

Since σ interchanges e_1 and e_4, it follows that $\langle x_2, x_3 \rangle \subseteq \operatorname{ann} M(e_1) = \operatorname{ann} M(e_4)$. But $\operatorname{GKdim}(A/\langle x_2, x_3 \rangle) = 1$, and therefore, to show equality, we need only show $\langle x_2, x_3 \rangle$ is prime. As (c) follows from the fact that σ fixes e_2 and e_3, the proposition would be proved if we show $\langle x_3 \rangle$ and $\langle x_2, x_3 \rangle$ are prime in A. We prove these in the following lemma. ■

Lemma 4.8. *Suppose σ interchanges the two rulings on Q. Then the ideals $\langle x_2 \rangle$, $\langle x_3 \rangle$, $\langle x_2, x_3 \rangle$ are prime (but not completely prime) in A.*

Note . When σ preserves the rulings on Q these ideals are not prime in A, since x_1, x_4 are normal and $x_1 A x_4 \subseteq \langle x_i \rangle$ for $i = 2, 3$.

Proof. We first show $\langle x_2, x_3\rangle$ is prime. By the proof of 1.8, or by 1.3(b), the ring $A/\langle x_2, x_3\rangle$ is isomorphic to the ring $R/\langle x^2+y^2\rangle$ where $R := k[x,y]$ has the defining relation $xy + yx = 0$. Thus R is a quantum plane whose point modules are parametrized by $\mathbb{P}^1$ with associated automorphism ([ATV1]) $\sigma_1 \colon \mathbb{P}^1 \to \mathbb{P}^1$ given by $\sigma_1(\alpha,\beta) = (\alpha, -\beta)$. The ideal $\langle x^2+y^2\rangle$ is the annihilator of the point module $M(1,1)$ and hence is prime in R.

Similarly, $A/\langle x_3\rangle$ is isomorphic to the ring $S/\langle x^2+z^2\rangle$ where $S := k[x,y,z]$ has the three defining relations $\gamma zy = yz$, $yx = -\gamma xy$, $xz = -zx$ where $\gamma \neq 0$. The point modules over S are parametrized by $\mathbb{P}^2$ with associated automorphism ([ATV1]) $\sigma_2 \colon \mathbb{P}^2 \to \mathbb{P}^2$ given by $\sigma_2(p_1,p_2,p_3) = (-\gamma p_1, p_2, \gamma p_3)$. We will show $\langle x^2+z^2\rangle$ is the annihilator of a line module for S, and hence is a prime ideal.

Consider the line $\ell = \mathcal{V}(x-z)$. From the above result for R, and the fact that $x^2+z^2 = (x-z)^2$ is normal in S, it follows that

$$\left\langle x^2+z^2\right\rangle \subseteq \operatorname{ann} M(\ell) \subsetneq \operatorname{ann} M(1,0,1) = \left\langle y, x^2+z^2\right\rangle. \qquad (*)$$

We claim $\langle x^2+z^2\rangle = \operatorname{ann} M(\ell)$. To prove this it suffices, by $(*)$, to show that $\langle y\rangle \cap \operatorname{ann} M(\ell) \subset \langle x^2+z^2\rangle$. Let $f \in \langle y\rangle \cap \operatorname{ann} M(\ell)$, and choose such an f to be nonzero of minimal y degree (which is possible as S has a PBW basis). Since y is normal, there exists $g \in S$ such that $f = gy$. However, $\operatorname{ann} M(\ell)$ is prime, so $g \in \operatorname{ann} M(\ell)$ (because y is normal). Thus, there exist $g_1, g_2 \in S$ such that $g = g_1 y + g_2(x^2+z^2)$ and $\deg_y(g_1 y) \le \deg_y(g) < \deg_y(f)$. But $g_1 y \in \langle y\rangle \cap \operatorname{ann} M(\ell)$ and hence, by choice of f, it follows that $g_1 y = 0$. Therefore, $g \in \langle x^2+z^2\rangle$, whence $f \in \langle x^2+z^2\rangle$, and this proves the claim. We conclude that $\langle x_3\rangle$ is prime in A. ■

We next consider the lines $\ell \not\subset Q$ that meet L and the points on Q that do not lie on the four special lines. Here the behaviour varies depending on the algebra (or rather, on σ) even if we restrict to the case where σ preserves the rulings on Q. To demonstrate this, we complete the section by contrasting examples 1.5 and 1.7.

Example 4.9. Suppose $A = \mathcal{O}_{\text{II}}(\mathcal{M}_{\in}(\mathbb{C}))$, where q is not a root of unity; or more generally, suppose $\alpha\lambda = 1$ in 1.3(a) with $|\langle\sigma\rangle \cdot p| = \infty$ for all $p \in Q \setminus \{e_j\}_{j=1}^4$. Then every line module has nonzero annihilator.

Note. The assumption $\alpha\lambda = 1$ implies x_1, x_4 are eigenvectors for τ corresponding to the same eigenvalue, and since they both vanish on L, any linear combination of them will be normal in A (by 1.11(d)).

Proof. Consider a line module, $M(\ell)$, for some line ℓ in $\mathbb{P}^3$. If $\ell \subset Q$ then, by 4.4, $\Omega \in \operatorname{ann} M(\ell)$. If $\ell \not\subset Q$, then ℓ meets L, so there exists $(\mu,\nu) \in \mathbb{P}^1$ such that $\ell \subset \mathcal{V}(\mu x_1 - \nu x_4)$. However, for all $(\mu,\nu) \in \mathbb{P}^1$ the element $\mu x_1 - \nu x_4$ is normal. It follows that $A(\mu x_1 - \nu x_4) \subseteq \operatorname{ann} M(\ell)$. ■

Moreover, $\langle \mu x_1 - \nu x_4\rangle$ is the annihilator of the plane module $A/A(\mu x_1 - \nu x_4)$, and so must be prime. If $\ell \neq L$ and $\ell \not\subset Q$, and if there exists $p \in \ell$ with $|\langle\sigma\rangle \cdot p| = \infty$ then, in fact, $\operatorname{ann} M(\ell) = \langle \mu x_1 - \nu x_4\rangle$ (by a similar GKdimension argument as used above combined with 4.4); in addition, if $p \in Q \setminus \{4 \text{ special lines}\}$ (i.e., if $\mu\nu \neq 0$) then $\operatorname{ann} M(p) = \langle \Omega, \mu x_1 - \nu x_4\rangle$ (since a basis argument shows $A/\langle \Omega, \mu x_1 - \nu x_4\rangle$ is a domain).

Example 4.10. Suppose $A = \mathcal{O}_{\text{II}}(\mathfrak{sp}\mathbb{C}^4)$. (Then $\alpha\lambda = q^{-2} \neq 1$.) If q is not a root of unity, then $\operatorname{ann} M(\ell) = 0$ for all lines $\ell \not\subset \mathcal{V}(x_1) \cup \mathcal{V}(x_4) \cup Q$.

Note. Here x_2, x_3 are eigenvectors for τ corresponding to the same eigenvalue, but neither x_2 nor x_3 vanishes identically on L.

Proof. For q not a root of unity, a "basis argument" proves that the prime ideals of A are:

(a) $\langle 0\rangle$,
(b) $\langle \Omega\rangle$, $\langle x_1\rangle$, $\langle x_4\rangle$,
(c) $\langle x_2+\gamma x_3\rangle$ for all $0\neq\gamma\in\mathbb{C}$,
$\langle x_1, x_3+\delta x_2x_4^2\rangle$,
$\langle x_4, x_2+\delta x_1^2x_3\rangle$ for all $\delta\in\mathbb{C}$,
$\langle x_i, x_j\rangle$ for $i<j$, $(i,j)\neq(2,3)$,
(d) $\langle x_i, x_j, x_m\rangle$ for i,j,m distinct,
(e) $\langle x_i, x_j, x_l, x_m-\gamma\rangle$ for i,j,l,m distinct and for all $\gamma\in\mathbb{C}$.

The maximal ideals are those in (e); the primitives are those in (a), (c), (e); types (b) and (d) are not primitive; and all the prime ideals are completely prime. Notice that $\Omega\in\langle\delta x_2+\gamma x_3\rangle$ for all $(\delta,\gamma)\in\mathbb{P}^1$, and that when $\gamma\neq 0$ we have $\langle x_2+\gamma x_3\rangle=\text{ann}M(p)$ for all $p\in\mathcal{V}(x_2+\gamma x_3)\cap Q$ which satisfy $|\langle\sigma\rangle\cdot p|=\infty$. If $\langle x_1\rangle\subseteq\text{ann}M(\ell)$ for some line module $M(\ell)$, then $\ell\subset\mathcal{V}(x_1)$ since $M(\ell)\cong A/Ax_1+Av$ for some $v\in A_1$. If, in addition, $\ell\neq L$ then the list of primes shows that equality must hold, that is, $\langle x_1\rangle=\text{ann}M(\ell)$. Similarly, $\langle x_4\rangle=\text{ann}M(\ell)$ for $\ell\subset\mathcal{V}(x_4)$, $\ell\neq L$. This contrasts with lines $\ell\not\subset\mathcal{V}(x_1)\cup\mathcal{V}(x_4)\cup Q$ for which the list implies $\text{ann}M(\ell)=0$. ■

An observation that arises from this last example is that (when q is not a root of unity) then every non-maximal homogeneous prime ideal of $\mathcal{O}_{\text{II}}(\mathfrak{sp}\mathbb{C}^4)$ is in fact the annihilator of some point, line (or plane) module.

5. Point Modules as Quotients of Line Modules

Recall the situation in §2.3 where a point module $M(p)$ is a quotient of a line module $M(\ell)$, and the kernel K was shown to be a shifted line module generated by an element of degree one; that is, we have a short exact sequence

$$0\longrightarrow M(\ell')[-1]\longrightarrow M(\ell)\longrightarrow M(p)\longrightarrow 0$$

where ℓ' is the line corresponding to $K\cong M(\ell')[-1]$. In this section we identify such lines ℓ' in terms of p and ℓ.

Remark 5.1. [A, corollary 2.10] Suppose that non-isomorphic point modules $M(p)$ and $M(q)$ are quotients of a line module $M(\ell)$. Consider the diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M(\ell')[-1] & \longrightarrow & M(\ell) & \longrightarrow & M(p) & \longrightarrow & 0\\ & & & & \big\downarrow & & & & \\ & & & & M(q) & & & & \end{array}$$

If the composition $M(\ell')[-1]\to M(q)$ is zero then $M(q)$ is a quotient of $M(p)$. But since both $M(q)$ and $M(p)$ have the same Hilbert series it follows that $M(q)\cong M(p)$, which is a contradiction. Therefore, the composition is a nonzero map $M(\ell')[-1]\to M(q)$, with image $M(q)_{\geq 1}\cong M(\sigma^{-1}(q))[-1]$. Hence $\sigma^{-1}(q)\in\ell'$.

Notation . The symbol ℓ_{pq} will denote the line through the points p and q.

Lemma 5.2. *Let ℓ be a line in $\mathbb{P}^3$ that contains at least three distinct points p, q, r $\in Q\cup L$. Then there is a short exact sequence*

$$0\longrightarrow M(\ell')[-1]\longrightarrow M(\ell)\longrightarrow M(p)\longrightarrow 0$$

where $\ell'=\ell_{\sigma^{-1}(q)\,\sigma^{-1}(r)}$.

Note . If ℓ in 5.2 is the distinguished line L then $\ell' = L$. On the other hand, if $\ell \subset Q$ then $\ell' = \sigma^{-1}(\ell)$.

Proof. The result follows from remark 5.1. ■

Recall the definition of τ from 1.10.

Lemma 5.3. *Let ℓ be a line in $\mathbb{P}^3$ that meets the distinguished line L at a point p. Then there is a short exact sequence*

$$0 \longrightarrow M(\tau^{-1}(\ell))[-1] \longrightarrow M(\ell) \longrightarrow M(p) \longrightarrow 0.$$

Proof. The result for $\ell = L$ follows from 5.2. Henceforth, assume $\ell \neq L$. Then we can assume $\ell = \mathcal{V}(u, v)$ such that $u, v \in A_1$ with $u(L) \neq 0$ and $v(L) \neq 0$. Since $p \in L$, there exists $w \in A_1$ such that $\{p\} = \mathcal{V}(u, v, w)$ and $w(L) = 0$. As w vanishes on L, it follows from 1.11(d) that $u^\tau w - w^\tau u = 0 = v^\tau w - w^\tau v$. Let $\bar{w}$ denote the image of w in $M(\ell) \cong A/Au + Av$. Then, by §2.3, $A\bar{w}$ is the kernel of the map $M(\ell) \to M(p)$ and the previous work shows that $u^\tau \bar{w} = 0 = v^\tau \bar{w}$. From linear independence of u and v, and bijectivity of τ, it follows that $M(\ell') \cong A/Au^\tau + Av^\tau$, which completes the proof. ■

Lemma 5.4. *Let ℓ be a line in $\mathbb{P}^3$ that meets $Q \cup L$ in exactly two points, $p \in Q \setminus L$ and $q \in L \setminus Q$. Then there is a short exact sequence*

$$0 \longrightarrow M(\ell')[-1] \longrightarrow M(\ell) \longrightarrow M(p) \longrightarrow 0$$

where $\ell' = \ell_{\sigma^{-1}(p)\,\sigma^{-1}(q)}$.

Proof. By remark 5.1, $\sigma^{-1}(q) \in \ell'$. We will show $\sigma^{-1}(p) \in \ell'$ also.

By hypothesis $\ell \not\subset Q$ and so, by 4.4, the normal element $\Omega \notin \operatorname{ann} M(\ell)$. Therefore, by [LS, 2.10], Ω is a nonzero divisor in $M(\ell)$ which implies $\Omega M(\ell)$ is isomorphic to a shifted line module. As $p \in Q$ we have $\Omega M(p) = 0$, whence $\Omega M(\ell) \subset K$ where K is the kernel of the map $M(\ell) \to M(p)$. Therefore, $K/\Omega M(\ell)$ is isomorphic to a shifted point module $M(r)[-1]$. Since $\Omega M(r) = 0$ and $M(r)$ is isomorphic to a quotient of $M(\ell')$, it follows that $r \in Q \cap \ell'$. Thus $\ell' = \ell_{r\,\sigma^{-1}(q)}$.

Next, write $\ell = \mathcal{V}(u, v)$ and $p = \mathcal{V}(u, v, w)$ where u, v, $w \in A_1$ and $u(L) = 0$. Then, as in the proof of 5.3, $K = A\bar{w}$ and $u^\tau \bar{w} = 0$. Moreover, by 1.11(c), $v^\tau w - w^\tau v \in k\Omega$, so that $v^\tau \bar{w} = 0$ in $K/\Omega M(\ell)$. Hence, $u^\tau(r) = 0 = v^\tau(r)$. Therefore, $\tau(r) \in \ell \cap Q = \{p\}$, giving $r = \tau^{-1}(p) = \sigma^{-1}(p)$. ■

The only type of line ℓ left to consider is one like ℓ_{pq} in 5.4, except assuming now that the point q belongs to $Q \cap L$. Unfortunately, the proof in 5.4 no longer works (indeed the result is different in general) as now $K/\Omega M(\ell) \cong M(q)[-1]$. The reader should compare the next two results with [ATV2, 6.28].

Lemma 5.5. *Let $\ell := \mathcal{V}(u, v)$ be a line in $\mathbb{P}^3$, where $u, v \in V$. If ℓ meets $Q \cup L$ in exactly two points, $p \in Q \setminus L$ and $q \in Q \cap L$, then there is a short exact sequence*

$$0 \longrightarrow M(\ell')[-1] \longrightarrow M(\ell) \longrightarrow M(p) \longrightarrow 0$$

where $\tau^{-1}\ell' = \mathcal{V}(\Omega u \Omega^{-1},\, \Omega v \Omega^{-1})$.

Proof. As in the proof of 5.4, $K/\Omega M(\ell) \cong M(r)[-1]$ for some $r \in Q$. We claim that $r = q$. Assuming this holds, we have a short exact sequence

$$0 \longrightarrow \Omega M(\ell)[1] \longrightarrow K[1] \cong M(\ell') \longrightarrow M(q) \longrightarrow 0$$

and $\Omega M(\ell)[1] \cong M(\ell_\Omega)[-1]$ for some line ℓ_Ω. Applying lemma 5.3 to this sequence, it follows that $\ell_\Omega = \tau^{-1}\ell'$. Since A is a domain and Ω is normal, there is a linear map $\theta \in$

$\mathrm{Aut}(A_1)$ such that $\Omega u = u^\theta \Omega$ and $\Omega v = v^\theta \Omega$. In $M(\ell)$ this translates to $u^\theta \bar{\Omega} = v^\theta \bar{\Omega} = 0$ where $\bar{\Omega}$ is the image of Ω in $M(\ell) \cong A/Au + Av$. Thus

$$M(\ell_\Omega) \cong \Omega M(\ell)[2] \cong \frac{A}{Au^\theta + Av^\theta}.$$

Hence, $\ell_\Omega = \mathcal{V}(u^\theta, v^\theta) = \theta^{-1}(\ell)$ which we write as $\mathcal{V}(\Omega u \Omega^{-1},\ \Omega v \Omega^{-1})$ (cf. [ATV2, 6.28]).

It remains to show that the point r found above is the given point q. Denoting the two lines on Q passing through p by ℓ_1, ℓ_2, and those on Q through q by ℓ_3, ℓ_4, we define $w, a \in A_1$ to be linear forms whose divisors of zeros on Q are $(w)_0 = \ell_1 + \ell_2$ and $(a)_0 = \ell_3 + \ell_4$. By the hypotheses on p, q, all the lines ℓ_i are distinct, and, relabelling if necessary, $\ell_2 \cap \ell_3 \neq \emptyset$. Next we take $v, b \in A_1$ with respective divisors of zeros on Q to be $(v)_0 = \ell_2 + \ell_3$ and $(b)_0 = \ell_1 + \ell_4$. Then $\ell = \mathcal{V}(v, b)$ and $w \notin kv \oplus kb$. Since $\ell \neq L$ we can assume $L \not\subset \mathcal{V}(v)$ (by relabelling v, b if necessary) and there exists $u \in kv \oplus kb$ such that $L \subset V(u)$ (possibly $u = b$). In summary, we have

$$\ell = \mathcal{V}(u, v), \quad p = \mathcal{V}(u, v, w), \quad M(\ell) \cong \frac{A}{Au + Av}, \quad K = A\bar{w}$$

where $\bar{w}$ denotes the image of w in $M(\ell)$. By construction

$$u^\tau w - w^\tau u = 0 \quad \text{in } A, \qquad v^\tau w - w^\tau v \in k\Omega, \tag{1}$$

and $(v^\tau \otimes b - a^\tau \otimes w)(\sigma^{-1}(s), s) = 0$ for all $s \in \cup \ell_i$. Furthermore, as in the proof of 2.8, multiplying a by a suitable scalar if necessary, we can conclude the latter holds for all $s \in Q$. Therefore

$$v^\tau b - a^\tau w \in k\Omega. \tag{2}$$

Combining (1) and (2) we find $u^\tau \bar{w} = v^\tau \bar{w} = a^\tau \bar{w} = 0$ in $K/\Omega M(\ell)$, proving that $r = \mathcal{V}(u^\tau, v^\tau, a^\tau) = q$. ∎

Corollary 5.6. (to previous proof).
Let $\ell := \mathcal{V}(u, v)$ be a line in $\mathbb{P}^3$ where $u, v \in V$. If $\ell \not\subset Q$ then $\Omega M(\ell) \cong M(\ell')[-2]$ where $\ell' = \mathcal{V}(\Omega u \Omega^{-1},\ \Omega v \Omega^{-1})$. ∎

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