Non-commutative Spaces for Graded Quantum Groups and Graded Clifford Algebras

Michaela Vancliff*
Department of Mathematics, Box 19408
University of Texas at Arlington
Arlington, TX 76019-0408
vancliff@uta.edu
www.uta.edu/math/vancliff.htm

Abstract. We propose that the notion of "quantum space" from Artin, Tate and Van den Bergh's non-commutative algebraic geometry be considered the "non-commutative space" of a quantum group.

1. Introduction

Analysis of solutions to the quantum Yang-Baxter equation from the quantum inverse scattering method is best approached via the irreducible finite-dimensional representations of the coordinate ring of a quantum group ([11, 12, 13, 35]). For this reason and many others, it is now a well-established fact that quantum groups are an important algebraic tool in mathematical physics.

In the spirit of classical physics, a quantum group is commonly viewed as a "deformation" of an algebra of functions of some variety or topological group, and so the quantum group is considered to be an algebra of non-commuting functions acting on some "non-commutative space" ([8]). However, at this time, there is still much debate over whether or not such a non-commutative space exists although it is believed that if it does, then it should be useful in determining properties of the quantum group and its representation theory. The objective of this paper is to demonstrate that a good candidate for this non-commutative space is the

*Supported in part by NSF grant DMS-9996056.

1991 *Mathematics Subject Classification.* 16W50, 14A22, 17B37, 81R50.

Key Words and Phrases. Regular algebra, quadratic algebra, point module, line module, point scheme, line scheme, quantum group, Clifford algebra, Poisson algebra, symplectic leaves.

notion of "quantum space" that arises in the theory of non-commutative algebraic geometry à la M. Artin, J. Tate and M. Van den Bergh ([1, 2, 3, 4]).

Roughly speaking, Artin, Tate and Van den Bergh use the category of graded modules of a non-commutative algebra as the space in which to do geometry. The first main success at applying this theory to quantum groups was seen in the analysis by S. P. Smith et al. in their classification of the finite-dimensional, irreducible representations of the Sklyanin algebra on four generators ([21, 32, 33, 31]). It should be emphasized that many physicists and mathematicians had attacked this problem since 1981, but without success until Smith et al. succeeded in 1991. Smith et al. used properties of the quantum space of the Sklyanin algebra in order to determine structural properties of the algebra. An attractive feature of this geometric theory is that it recovers commutative algebro-geometric results in addition to being applicable to non-commutative algebras.

In Sections 2-4, some examples from quantum groups will be discussed via these geometric ideas; namely, the Sklyanin algebra, the coordinate ring of quantum matrices and graded Clifford algebras. In Section 5, the theory of non-commutative algebraic geometry will be outlined in general with recent developments also discussed.

Surprisingly, within the context of Poisson algebras and quadratic quantum groups, the points of the quantum space of the quantum group are related to the projective zero-dimensional symplectic leaves of the Poisson manifold (Theorem 6.18).

Recently, the theory has begun to grow independently of deformational issues, with a shift in emphasis towards analyzing non-commutative projective schemes in their own right ([6, 14, 34, 42, 43]). For example, in [42], Van den Bergh has developed a theory of non-commutative blowing up at a point on a non-commutative projective surface in order to classify the finite-dimensional irreducible representations of the family of three-dimensional regular algebras in [2, 3]. Thus, an intersection theory and a notion of isomorphism between non-commutative geometric objects is needed in order to compare geometric objects in one "quantum space" with those in another and this is the current focus of P. Jorgensen, S. P. Smith and J. Zhang in [14, 34].

Throughout the paper, k denotes an algebraically closed field such that $\text{char}(k) \neq 2$ and $\mathcal{V}(X)$ denotes the locus of common zeros of the set X.

2. The Sklyanin Algebra

In the early 1980s, using Baxter's elliptic solutions to the Yang-Baxter equation, Sklyanin constructed a family of graded algebras on four generators defined in terms of elliptic functions ([29]). In [10], these algebras (now dubbed Sklyanin algebras) were shown to depend on an elliptic curve in $\mathbb{P}^3$ and an automorphism. In this section, we discuss the Sklyanin algebra on three generators and the Sklyanin algebra on four generators.

It should be noted that methods which had been effective for enveloping algebras, group algebras of polycyclic-by-finite groups and polynomial-identity rings were ineffective when applied to the Sklyanin algebras, mainly because they have nothing like a Poincaré-Birkhoff-Witt basis.

2.1 The Sklyanin Algebra on Three Generators

Let $a, b, c \in k^\times$ such that $(3abc)^3 \neq (a^3 + b^3 + c^3)^3$. The k-algebra A on generators x, y and z with defining relations

$$cx^2 + bzy + ayz = 0,$$
$$cy^2 + bxz + azx = 0,$$
$$cz^2 + byx + axy = 0,$$

is the Sklyanin algebra on three generators ([2, 3]). Consider these relations as bihomogeneous $(1, 1)$-forms on $\mathbb{P}^2 \times \mathbb{P}^2$, and let Γ denote their locus of common zeros in $\mathbb{P}^2 \times \mathbb{P}^2$. It is shown in [3] that Γ is the graph of an automorphism σ of an elliptic curve E in $\mathbb{P}^2$, where $E = \mathcal{V}(\,(a^3 + b^3 + c^3)xyz - abc(x^3 + y^3 + z^3)\,)$. The following is proved in [3, 4].

A *twisted homogeneous coordinate ring* $B(E)$ of E is obtained by using the automorphism σ to deform the multiplication in the commutative homogeneous coordinate ring of E. We refer the reader to [3, 5] for details. This process produces a non-commutative algebra $B(E)$ whose algebraic properties are entwined in the geometric properties of E. The algebra $B(E)$ is a quotient of A via $B(E) = A/\langle g \rangle$ where $g \in A$ is homogeneous of degree three. As such, the algebraic properties of A are encoded in E and σ, and this is demonstrated by the fact that g belongs to the centre of A. Indeed, A is a finite module over its centre if and only if σ has finite order.

The geometry determined by E and σ were important tools in establishing that A is a noetherian domain with Hilbert series the same as that of the polynomial ring on three generators.

The above discussion suggests that the elliptic curve E together with its automorphism σ should play the role of a non-commutative space for A. In fact, Section 4 will demonstrate that sometimes the locus of zeros of the defining relations is too small to be tractable. The notion of "quantum space" (to be discussed in Section 5) for A includes E and σ plus some other geometric data in such a way that the points of E may be viewed as points of the quantum space.

2.2 The Sklyanin Algebra on Four Generators

The Sklyanin algebra on four generators is the k-algebra A on generators x_0, x_1, x_2 and x_3 with six defining relations

$$\begin{aligned} x_0x_i - x_ix_0 &= \alpha_i(x_jx_k + x_kx_j), \\ x_0x_i + x_ix_0 &= x_jx_k - x_kx_j, \end{aligned}$$

where (i, j, k) is a cyclic permutation of $(1, 2, 3)$, and $(\alpha_1, \alpha_2, \alpha_3) \in k^3$ where $\alpha_i \neq 0, \ \pm 1$ for all i and $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_1\alpha_2\alpha_3 = 0$ ([29]). The reader is referred to [21, 32, 33] for details of the following.

Consider these relations as bihomogeneous $(1, 1)$-forms on $\mathbb{P}^3 \times \mathbb{P}^3$. Their locus of common zeros $\Gamma \subset \mathbb{P}^3 \times \mathbb{P}^3$ is the graph of an automorphism σ of a subscheme $E \cup S \subset \mathbb{P}^3$, where E is an elliptic curve and S consists of the four points

$$e_1 = (1, 0, 0, 0), \quad e_2 = (0, 1, 0, 0), \quad e_3 = (0, 0, 1, 0) \quad \text{and} \quad e_4 = (0, 0, 0, 1)$$

in $\mathbb{P}^3$.

As in the three-generator case, one may form the twisted homogeneous coordinate ring $B(E)$ of E by using the automorphism σ to deform the multiplication of the commutative homogeneous coordinate ring of E. As before, the algebra $B(E)$ is a quotient of A via $B(E) = A/\langle g_1, g_2\rangle$ where $g_1, g_2 \in A$ are homogeneous of degree two. In fact, $kg_1 \oplus kg_2$ belongs to the centre of A, and generates the centre of A if σ has infinite order. Moreover, A is a finite module over its centre if and only if σ has finite order. As before, the geometry of Γ establishes that A is a noetherian domain with Hilbert series the same as that of the polynomial ring on four generators.

The abelian group structure of the elliptic curve E and the position of the e_i in $\mathbb{P}^3$ with respect to E were critical components in conclusively determining the finite-dimensional irreducible representations of A ([31, 33]).

The notion of "quantum space" (to be discussed in Section 5) for A includes $E \cup S$ and

σ plus some other geometric data in such a way that the points of $E \cup S$ may be viewed as points of the quantum space.

3. The Coordinate Ring of Quantum Matrices

Let M_n denote the ring of $n \times n$ matrices over k and, for $q \in k^\times$, let $\mathcal{O}_q(M_n)$ denote the coordinate ring of quantum $n \times n$ matrices over k ([9]). The algebra $\mathcal{O}_q(M_n)$ is the k-algebra on n^2 generators $\{x_{ij} : 1 \leq i, j \leq n\}$, with defining relations determined by the requirement that whenever $r < s$ and $l < m$ there exists a k-algebra isomorphism:

$$k_q \begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{\sim} k_q \begin{bmatrix} x_{rl} & x_{rm} \\ x_{sl} & x_{sm} \end{bmatrix},$$

where the k-algebra $k_q[a, b, c, d] = k_q \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has the six defining relations:

$$\begin{array}{rclrclrcl} ab & = & qba, & bd & = & qdb, & bc & = & cb, \\ cd & = & qdc, & ac & = & qca, & ad - da & = & (q - q^{-1})bc. \end{array}$$

The results stated in this section are given in [38].

We first focus on the case $n = 2$. As in Section 2.2, consider the relations of $\mathcal{O}_q(M_2)$ as bihomogeneous $(1,1)$-forms on $\mathbb{P}^3 \times \mathbb{P}^3$. If $q^2 \neq 1$, then the locus of common zeros in $\mathbb{P}^3 \times \mathbb{P}^3$ of the relations is the graph of an automorphism σ of a subscheme $Q \cup L$ of $\mathbb{P}^3$, where $Q = \mathcal{V}(ad - bc)$ is a nonsingular quadric in $\mathbb{P}^3$ corresponding to the matrices with zero determinant, and $L = \mathcal{V}(b, c)$ is a line in $\mathbb{P}^3$ which meets Q at two distinct points and corresponds to the diagonal matrices.

There is a unique nonzero homogeneous element (up to nonzero scalar multiples) of degree two in $\mathcal{O}_q(M_2)$ which vanishes on the graph of $\sigma|_Q$ but not on the graph of σ. This distinguished element Ω is the famous quantum determinant of $\mathcal{O}_q(M_2)$ which is central in $\mathcal{O}_q(M_2)$. In fact, $\mathcal{O}_q(M_2)/\langle\Omega\rangle$ is the twisted homogeneous coordinate ring of Q determined by $\sigma|_Q$. Moreover, the twisted homogeneous coordinate ring of the line L determined by $\sigma|_L$ is given by $\mathcal{O}_q(M_2)/\langle b, c\rangle$. In other words, it is as if the geometry is determining the quantum determinant and the normal elements b and c in $\mathcal{O}_q(M_2)$.

Since $\mathcal{O}_q(M_2)$ is a finite module over its centre if and only if q has finite order, it follows from [38] that $\mathcal{O}_q(M_2)$ is a finite module over its centre if and only if σ has finite order.

The case $n > 2$ generalizes that of $n = 2$ as follows. For all $n \geq 2$ and for all $q \in k^\times$, the locus of common zeros of the defining relations of $\mathcal{O}_q(M_n)$ is the graph of an automorphism of a subscheme of $\mathbb{P}(M_n)$. If $q^2 \neq 1$, this subscheme is independent of q and is the nondisjoint union of $\binom{n}{2}^2$ copies of $Q \cup L$ and $\binom{n}{d}^2$ copies of $\mathbb{P}^{d-1}$ for all $d = 1, \ldots, n$. As in the case $n = 2$, the quantum determinant of $\mathcal{O}_q(M_n)$ may be read off from this subscheme and its automorphism and so can the normal elements of degree one in $\mathcal{O}_q(M_n)$. The notion of "quantum space" (to be discussed in Section 5) for $\mathcal{O}_q(M_n)$ includes this subscheme together with its automorphism, plus some other geometric data in such a way that the points of this subscheme may be viewed as points of the quantum space.

4. Graded Clifford Algebras

Let $R = k[y_1, \ldots, y_n]$ denote the commutative polynomial ring on n variables and let $Y = (Y_{ij}) \in M_n(R)$ denote a symmetric matrix whose entries are homogeneous linear polynomials in the variables y_i.

Definition 4.1 In the terminology of [41] and [17, §4], the graded *Clifford algebra* $A = A(Y)$ over R associated to Y is the k-algebra on generators $x_1, \ldots, x_n$, $y_1, \ldots, y_n$ with defining relations $x_i x_j + x_j x_i = Y_{ij}$ for all i, j, and y_i central for all i. The algebra A is graded by taking $\deg(x_i) = 1$ and $\deg(y_i) = 2$ for all i.

Writing $Y = Y_1 y_1 + \cdots + Y_n y_n$, where the $Y_i \in M_n(k)$ are symmetric matrices, we may associate to Y an n-dimensional linear system $\mathcal{Q} = kQ_1 + \cdots + kQ_n$ of quadrics $Q_1, \ldots, Q_n \subset \mathbb{P}^{n-1}$ by taking each Q_i to be the quadric in $\mathbb{P}^{n-1}$ corresponding to Y_i; that is, $Q_i = \{y \in \mathbb{P}^{n-1} : y^T Y_i y = 0\}$. A base point of $\mathcal{Q}$ is a common point of intersection of all the Q_i.

If $\mathcal{Q}$ has no base points, then A is generated by the x_i only and is a finite module over R and hence noetherian. In this case, by [7, §3], A is the enveloping algebra of a Lie superalgebra and so has $n(n-1)/2$ defining relations and they are of the form $\sum_{ij} \alpha_{ijm}(x_i x_j + x_j x_i) = 0$ where the $\alpha_{ijm} \in k$, and A has Hilbert series the same as that of the polynomial ring on n variables. Henceforth, we assume that $\mathcal{Q}$ has no base points.

We write $A = k\langle x_1, \ldots, x_n \rangle / \langle W \rangle$ where W is the span of the defining relations of A. As in the preceding sections, consider the relations as bihomogeneous $(1,1)$-forms on $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$ and let Γ denote their locus of common zeros. Let $\pi_i : \Gamma \to \mathbb{P}^{n-1}$ denote the ith projection map. Since the defining relations of A have the aforementioned description, $\pi_1(\Gamma) = \pi_2(\Gamma)$. Owing to the symmetry of the defining relations, if $(a, b) \in \Gamma$, then so is (b, a); in other words, Γ is the graph of an automorphism $\sigma : \pi_1(\Gamma) \to \pi_1(\Gamma)$ where σ has order two.

In this situation, a point $(a, b) \in \Gamma$ if and only if $\sum_{ij} \alpha_{ijm}(a_i b_j + a_j b_i) = 0$ for all $m \leq n(n-1)/2$. In other words, $(a, b) \in \Gamma$ if and only if the symmetric matrix $ab^T + ba^T = (a_i b_j + a_j b_i)$ is a zero of $\sum_{ij} \alpha_{ijm} X_{ij}$ for all $m \leq n(n-1)/2$, where X_{ij} denotes the ijth coordinate function. Following [40], we consider symmetric $n \times n$ matrices as $\mathbb{P}^N$, where $N + 1 = n(n+1)/2$, and define a map $\phi : \mathbb{P}^{n-1} \times \mathbb{P}^{n-1} \to \mathbb{P}^N$ by $\phi(u, v) = uv^T + vu^T$ (which is a matrix of rank ≤ 2). Since k is algebraically closed with $\mathrm{char}(k) \neq 2$, and since $\mathrm{GL}_n(k)$ acts transitively on matrices of the same rank, the image of ϕ consists of all symmetric $n \times n$ matrices X such that $\mathrm{rank}(X) \leq 2$.

Depending on the defining relations, it is possible for Γ to be finite. In fact, by [40, Theorem 1.7], the cardinality of Γ is $2r_2 + r_1 \in \mathbb{N} \cup \{0, \infty\}$ where r_j denotes the number of matrices in $\mathbb{P}(\sum_{i=1}^{n} kY_i)$ which have rank j. If Γ is finite, then $r_1 \in \{0, 1\}$. The surprising fact is that it is possible for Γ to consist of only one point (counted with multiplicity twenty) ([40]). It is perhaps even more surprising that even if Γ consists of only one point, the space W consists of all those bihomogeneous $(1,1)$-forms which vanish on Γ ([26]). It follows that properties of the algebra are entwined in properties of Γ. In spite of this, one finds that algebraic properties of the algebra do not leap to the reader's eye from Γ as they seem to be doing in the preceding examples. This suggests that perhaps there is more geometric data that one may associate to A which, although completely determined by Γ, would be more insightful. The notion of "quantum space" (to be discussed in Section 5) for A includes Γ plus some other geometric data in such a way that the points of Γ may be viewed as points of the quantum space.

5. Non-commutative Algebraic Geometry

In the previous sections, the algebras were quadratic, and the locus Γ of common zeros of each algebra's defining relations was considered. In this section the notion of quantum space will be defined and the scheme Γ will constitute (some of the) points in the quantum space. Roughly speaking, the quantum space of an algebra is a (quotient) category of graded modules of the algebra, in which certain modules play the role of points, certain modules play the role of lines, and so forth. In addition, there are incidence relations between the modules which determine whether or not a point lies on a line, a line lies on a plane, etc. Within this category there are certain modules which are parametrized by projective schemes and may be determined via computation. Two such schemes are the so-called *point scheme* and *line scheme* of [28, 39].

Until Section 5.3, unless otherwise stated, A denotes a finitely generated, connected, positively graded k-algebra generated by homogeneous elements of degree one, and A_i denotes the homogeneous degree-i elements of A. Any A-module will be either a left A-module or a right A-module.

5.1 The Point Scheme

Definition 5.2 [3] A point module of A is a graded, cyclic A-module which is generated by elements of degree zero and which has Hilbert series $(1-t)^{-1}$. A truncated point module of length r is a graded, cyclic A-module which is generated by elements of degree zero and which has Hilbert series $1+t+\cdots+t^{r-1}$.

In particular, if M is a point module, then $M = \oplus_{i=0}^{\infty} M_i$ where $\dim_k(M_i) = 1$ for all i.

If A is commutative and generated by $n+1$ elements, then it is the homogeneous coordinate ring of a subscheme S of $\mathbb{P}^n$. In this case, any truncated point module of length three may be extended to a point module, and the set of point modules of A is in one-to-one correspondence with the points of S. In fact, if $M = AM_0$ is a point module, then $\mathcal{V}(\{a \in A_1 : aM_0 = 0\})$ is a point of S and $M \cong A/A(\{a \in A_1 : aM_0 = 0\})$. Conversely, if $p = \mathcal{V}(a_1, \ldots, a_n) \in S$, then $A/(Aa_1 + \cdots + Aa_n)$ is a point module.

Suppose A is non-commutative. If M is a truncated (left) point module of length three, then $\mathcal{V}(\{a \in A_1 : aM_0 = 0\})$ is a point of $\mathbb{P}^n$ and $A/A(\{a \in A_1 : aM_0 = 0\}) \twoheadrightarrow M$.

Theorem 5.3 [3] *There is a scheme which represents the functor of truncated point modules of length three.* ■

If A has the property that every truncated point module of length three may be extended to a point module, then this result says that the point modules are parametrized by a scheme (called the *point scheme* in [39]). Algebras of global dimension three which are "regular", according to the following definition, were considered in [2, 3, 4] and have this property.

Definition 5.4

1. [2, Page 171] The algebra A is called regular if the global (homological) dimension of A (gldim(A)) is finite, A has polynomial growth, and A is Gorenstein in the sense that $\mathrm{Ext}^q_A(k, A) = \delta^q_n k$ where $n = \mathrm{gldim}(A)$.

2. [20, §2] The algebra A is called Auslander-regular if gldim(A) is finite, and if, for every finitely generated A-module M and for every $i \geq 0$ and for every A-submodule N of $\mathrm{Ext}^i_A(M, A)$, we have $j(N) \geq i$, where
$$j(N) = \inf\{j : \mathrm{Ext}^j_A(N, A) \neq 0\}.$$

3. [20, §5] The algebra A is said to satisfy the Cohen-Macaulay property if GKdim(A) $= j(M)$+GKdim(M) for all nonzero finitely generated A-modules M where GKdim denotes Gelfand-Kirillov dimension.

In [20], it was shown that any Auslander-regular algebra which has polynomial growth is regular. The algebras considered in Sections 2-4 of this article are noetherian, Auslander-regular, have polynomial growth and satisfy the Cohen-Macaulay property ([2, 3, 4, 21, 32, 37, 38, 40]). If the global dimension is three, then regularity is sufficient to ensure that the zero locus of the defining relations is the graph of an automorphism, and in this case the point scheme is the zero locus of the defining relations ([3]).

The following result was proved in [39] with an additional geometric hypothesis which was subsequently removed in [27].

Theorem 5.5 [27] *Suppose that A is a quadratic, connected, finitely generated, noetherian k-algebra, which is Auslander-regular of global dimension four, and satisfies the Cohen-Macaulay property. If the Hilbert series of A is the same as that of the polynomial ring on four variables, then the zero locus Γ of the defining relations of A is the graph of an automorphism of a subscheme of $\mathbb{P}^3$. In particular, any truncated point module of length three may be extended to a point module, and Γ represents the functor of point modules, so it is the point scheme.* ■

These results on regularity suggest that noetherian (Auslander-)regular algebras of polynomial growth which satisfy the Cohen-Macaulay property are non-commutative analogues of the polynomial ring. Moreover, they suggest that to understand such algebras, one should first classify which subschemes and automorphisms arise as the zero locus of defining relations.

The regular algebras of global dimension three were classified and fully analyzed in [2, 3, 4]. However the regular algebras of global dimension four are not yet classified as this case appears to be less accessible; for instance, such an algebra's defining relations need not be determined by their zero locus Γ even if Γ is the point scheme ([27, 39, 44]). Instead, attention has shifted to *generic* regular algebras of global dimension four, in particular those which are quadratic on four generators with six defining relations. It is now well known that the relations of a quadratic algebra on four generators with six *generic* defining relations has zero locus consisting of twenty points. This motivates the classification of regular algebras of global dimension four which have a finite point scheme. Indeed, the following result demonstrates that even if a quadratic algebra is not regular, the defining relations may be recovered from their zero locus providing the latter is finite.

Theorem 5.6 [28] *Suppose A is a quadratic, connected, finitely-generated k-algebra on four generators with six defining relations. If the zero locus Γ of the defining relations of A is finite, then the span of the defining relations consists of those $(1,1)$-forms which vanish on Γ.* ■

This result motivates classifying the geometric data which arises as the zero locus Γ of the defining relations of a quadratic algebra. Recall from Section 4 that it is possible to construct graded Clifford algebras which have one point of multiplicity twenty as the zero locus of the defining relations. In that case, the point is the point scheme. It follows from Theorem 5.6 that the defining relations of such a Clifford algebra are completely determined by that one point (and automorphism) (see [26] for additional details).

5.2 The Line Scheme

It was remarked in Section 4 that sometimes the point scheme is too small to give much insight into the structure of the algebra, even if the point scheme determines the defining relations.

Definition 5.7 [3] A line module of A is a graded, cyclic A-module which is generated by elements of degree zero and which has Hilbert series $(1-t)^{-2}$. A truncated line module of length r is a graded, cyclic A-module which is generated by elements of degree zero and which has Hilbert series $1 + 2t + \cdots + rt^{r-1}$.

In particular, if M is a line module, then $M = \oplus_{i=0}^{\infty} M_i$ where $\dim_k(M_i) = i + 1$ for all i.

If A is commutative with $n + 1$ generators, then any truncated line module of length three may be extended to a line module, and the set of line modules of A is in one-to-one correspondence with the lines on the scheme S for which A is the homogeneous coordinate ring. Mimicking the discussion for point modules, if $M = AM_0$ is a line module, then $\mathcal{V}(\{a \in A_1 : aM_0 = 0\})$ is a line on S and $M \cong A/A(\{a \in A_1 : aM_0 = 0\})$. Conversely, if $\ell = \mathcal{V}(a_1, \ldots, a_{n-1})$ is a line on S, then $A/(Aa_1 + \cdots + Aa_{n-1})$ is a line module.

Suppose A is non-commutative. If M is a truncated (left) line module of length three, then $\mathcal{V}(\{a \in A_1 : aM_0 = 0\})$ is a line in $\mathbb{P}^n$ and $A/A(\{a \in A_1 : aM_0 = 0\}) \twoheadrightarrow M$.

Theorem 5.8 [28] *There is a scheme which represents the functor of truncated line modules of length three.* ■

If A is regular of global dimension three or if, instead, A is quadratic, noetherian, Auslander-regular of global dimension four, satisfies the Cohen-Macaulay property and has Hilbert series the same as that of the polynomial ring on four variables, then every truncated line module of length three may be extended to a line module ([3, 21]). In this case, Theorem 5.8 says that the line modules are parametrized by a scheme, the *line scheme*. It is straightforward to show that under these conditions, the line scheme is at least one-dimensional (see [40]). Moreover, if the line scheme has minimal dimension, then it determines the defining relations ([28]).

Even if the line scheme is well defined, it might not be straightforward to compute. The reader is referred to [28] for methods on computing the line scheme and for results on the line scheme of graded Clifford algebras which have a singleton point scheme.

5.3 Quantum Spaces

Let $M(p)$ denote a point module corresponding to a point p and let $M(\ell)$ denote a line module corresponding to a line ℓ. If $M(\ell) \twoheadrightarrow M(p)$, then $p \in \ell$. The converse holds if A is Auslander-regular of global dimension three or four. Thus, one may do geometry with the graded modules, where the incidence relations between the modules play the role of containment of the physical objects.

Of course, one may define higher dimensional linear modules, and truncated d-linear modules, in analogy with point and line modules. It is proved in [28] that there is a scheme which represents the functor of truncated d-linear modules.

In the commutative setting, this idea of doing geometry with modules in place of geometric objects was considered in the 1950s by J.-P. Serre as follows. Let B be a finitely generated, commutative, positively graded, connected k-algebra generated by B_1, and let gr-B denote the category of finitely generated, graded B-modules.

Theorem 5.9 [25] *The category of coherent sheaves on* Proj B *is equivalent to the quotient category* (gr-B)$/\mathcal{T}$*, where* $\mathcal{T}$ *denotes the full subcategory of* gr-B *of modules of finite length.*

Moreover, Proj B may be recovered from the quotient category (gr-B)$/\mathcal{T}$ as follows. The shift $M[m]$ of a module M, defined by $M[m]_n = M_{m+n}$, corresponds to tensor product by the polarizing invertible sheaf $\mathcal{L} = \mathcal{O}_{\text{Proj}B}(1)$ of linear forms on Proj B; that is, $\mathcal{M}[1] = \mathcal{M} \otimes \mathcal{L}$, where $\mathcal{M}$ is the coherent sheaf corresponding to M. It follows that this shift operation defines an autoequivalence on the quotient category (gr-B)$/\mathcal{T}$, so that the graded algebra $\bigoplus_{n=0}^{\infty} H^0(\text{Proj } B, \mathcal{L}^{\otimes n})$ is isomorphic to B in sufficiently high degree.

Inspired by Serre's theorem, M. Artin extended these ideas in [1] to the non-commutative setting as follows. Let A be a noetherian, positively graded, connected k-algebra generated by A_1, and let gr-A denote the category of finitely generated, graded A-modules.

Definition 5.10 [1] Define Proj A to be the triple $(\,(\text{gr-A})/\mathcal{T}, \mathcal{O}, \sigma)$ where $\mathcal{T}$ denotes the subcategory of gr-A of torsion modules, $\mathcal{O}$ denotes an object of (gr-A)$/\mathcal{T}$ which is represented by the left module A, and σ is the operation $\mathcal{M} \to \mathcal{M}[1]$ on (gr-A)$/\mathcal{T}$ induced by the shift of degree on an A-module. A quantum $\mathbb{P}^2$, or quantum projective plane, is Proj A where A is a regular algebra of global dimension three.

Given a quantum group A which satisfies the hypotheses of Definition 5.10, one may analyze Proj A and consider it the non-commutative space of the quantum group. The very nature of the quantum space of a quantum group is to produce certain graded modules which are tools for finding and classifying other modules. For instance, the finite-dimensional irreducible representations of the Sklyanin algebra on four generators are quotients of line modules ([33]). In general, a quantum (projective) space should be Proj A for some A, but it is unclear if regularity of A alone is sufficient to guarantee that A may be sensibly viewed as a deformation of a polynomial ring. As such, there is still debate over the definition of a quantum $\mathbb{P}^3$ and of a quantum projective space. However, the quantum groups in Sections 2-4 satisfy all the homological properties discussed herein, and so their associated category Proj are quantum spaces.

It should be emphasized that the point scheme and line scheme are subschemes of Proj A. In fact, d-linear modules play the role of d-dimensional linear objects. However, some noncommutative algebras have other graded modules which also play the role of d-dimensional linear objects and yet such modules are not equivalent to d-linear modules for any d. Although there is no obvious $\mathbb{P}^n$ in which these objects may be visualised (unlike linear modules), the presence of such modules means that even more geometric structure may be associated to A.

We remark that even if the algebra A is not regular, Proj A might still encode properties of A. This was demonstrated in [36, Chapter 2], where a quadratic algebra $A_{(4)}$ was produced from a procedure which is the quantum analogue of a classical construction as follows. Let $U_q(\mathfrak{sl}_2)$ denote the quantum universal enveloping algebra of $\mathfrak{sl}_2$ ([16, 30]). It is proved in [22, 23, 24] that if q is not a root of unity, then for each $n \in \mathbb{N}$ there are four non-isomorphic simple n-dimensional $U_q(\mathfrak{sl}_2)$-modules depending on a fourth root of unity ω and that every finite-dimensional $U_q(\mathfrak{sl}_2)$-module is semisimple. Let V_n denote either an n-dimensional simple $U_q(\mathfrak{sl}_2)$-module corresponding to "$\omega = 1$", or an n-dimensional simple $U(\mathfrak{sl}_2)$-module. Then $V_n \otimes V_n = W_1 \oplus W_3 \oplus \cdots \oplus W_{2n-1}$ where W_i is a simple i-dimensional module of $U_q(\mathfrak{sl}_2)$ (respectively, $U(\mathfrak{sl}_2)$). Let $A_{(n)} := T(V_n)/\langle U_n \rangle$ where $T(V_n)$ denotes the tensor algebra on V_n and U_n denotes an $n(n-1)/2$-dimensional submodule of $V_n \otimes V_n$; so, for example, $A_{(3)} = T(V_3)/\langle W_3 \rangle$ and $A_{(4)} = T(V_4)/\langle W_1 \oplus W_5 \rangle$. Classically, $A_{(n)} \cong S(V_n)$, the symmetric algebra on V_n; in particular, the classical $A_{(n)}$ is noetherian and regular of Gelfand-Kirillov dimension n.

In the quantum case, $A_{(3)} \cong \mathcal{O}_q(\mathfrak{so}k^3)$, the coordinate ring of quantum Euclidean three-dimensional space. The quantum $A_{(4)}$ is not so straightforward to understand. In particular, in [36], the quantum $A_{(4)}$ is shown not to be regular nor a domain. Nevertheless, its point scheme is well defined and is the graph of an automorphism σ of a twisted cubic curve C

in $\mathbb{P}^3$, even though the point scheme is not the locus of zeros of the defining relations. The structure of $A_{(4)}$ is determined by a nilpotent ideal N which is generated by the nonzero degree-two elements of $A_{(4)}$ which vanish on the graph of σ. The quotient algebra $A_{(4)}/N$ is a twisted homogeneous coordinate ring of C. Using this geometric data, it is straightforward to show that $A_{(4)}$ is noetherian of Gelfand-Kirillov dimension two (not four).

6. Quantum Spaces via Poisson Geometry

The setting in this section is that of a non-commutative quadratic deformation A of the polynomial ring which induces a Poisson bracket on the polynomial ring. Our main result, Theorem 6.18, states (essentially) that the point scheme is contained in the scheme of projective zero-dimensional symplectic leaves.

Example 6.11 If $q \in \mathbb{C}$ is generic, then $\mathcal{O}_q(M_2)$ from Section 3 induces a Poisson bracket on the polynomial ring $\mathcal{O}(M_2)$ on four variables via

$$\{b, c\} = 0, \quad \{a, d\} = 2bc, \quad \{a, b\} = ab, \quad \{a, c\} = ac, \quad \{b, d\} = bd, \quad \{c, d\} = cd,$$

and similarly for $\mathcal{O}_q(M_n)$ ([38]). Since the bracket is homogeneous, it lifts to $(\mathcal{O}(M_n)[z^{-1}])_0$, the degree-zero part of the localization $\mathcal{O}(M_n)[z^{-1}]$, for all nonzero, homogeneous $z \in \mathcal{O}(M_n)$. In this way, $\mathbb{P}(M_n)$ is equipped with a Poisson structure, and it is proved in [38] that if $q^2 \neq 1$, then the scheme of zero-dimensional symplectic leaves is precisely the first projection of the point scheme of $\mathcal{O}_q(M_n)$ to $\mathbb{P}(M_n)$.

There are many other quantum groups where the point scheme gives precisely the scheme of zero-dimensional symplectic leaves, such as the Sklyanin algebra on three generators, the coordinate ring, $\mathcal{O}_q(\mathfrak{so}\mathbb{C}^3)$, of quantum Euclidean three-dimensional space, and the coordinate ring, $\mathcal{O}_q(\mathfrak{sp}\mathbb{C}^{2n})$, of quantum symplectic $2n$-dimensional space, to name a few. However, the following example points out that equality need not hold in general.

Example 6.12 Let $\mathbb{C}[h]$ denote the polynomial ring in an indeterminate h, and let A denote the $\mathbb{C}[h]$-algebra on generators x, y and z with defining relations

$$xy = (1 + 2h)yx, \quad yz = (1 - h)zy \quad \text{and} \quad zx = (1 - h)xz,$$

so that $A(0) = A/\langle h\rangle$ is the polynomial ring on three variables. Moreover, $A(q) = A/\langle h - q\rangle$ is an iterated Ore extension for all $q \in \mathbb{C}$ such that $(1 + 2q)(1 - q) \neq 0$. A Poisson bracket is induced by A on $A(0)$ and is given by

$$\{x, y\} = 2xy, \quad \{y, z\} = -yz, \quad \{z, x\} = -xz$$

(see page 16). In this case, the scheme consisting of the projective zero-dimensional symplectic leaves is $\mathbb{P}^3$, but for all $q \in \mathbb{C}^\times$ such that $(1 + 2q)(1 - q) \neq 0$, the projection of the point scheme of $A(q)$ to its first component in $\mathbb{P}^3$ is $\mathcal{V}(xyz)$. So, for this example, the point scheme encodes the normal elements x, y and z of $A(q)$ and the Poisson ideals $\langle x\rangle$, $\langle y\rangle$ and $\langle z\rangle$ of $A(0)$ via the components of the point scheme, whereas the scheme of zero-dimensional symplectic leaves loses this information.

Our goal is to prove that in general the point scheme is contained in the scheme of zero-dimensional symplectic leaves.

Let $\mathbb{C}'$ denote a subset of $\mathbb{C}$ containing zero whose complement is a finite set. Let h be an indeterminate and let $\mathbb{C}(h)$ denote the ring of rational functions on $\mathbb{C}$, and let R denote $\mathbb{C}[h][\Pi_r(h-r)^{-1}] \subset \mathbb{C}(h)$ where r runs through a subset of $\mathbb{C} \setminus \mathbb{C}'$; in particular, R is a ring of rational functions on $\mathbb{C}$ which are defined on $\mathbb{C}'$ (or possibly more than $\mathbb{C}'$). For each $q \in \mathbb{C}'$, let $\mathfrak{m}_q$ denote the maximal ideal $R(h - q)$. Fix a finite-dimensional $\mathbb{C}$-vector space V and an R-submodule $W \subset R \otimes_{\mathbb{C}} V \otimes_{\mathbb{C}} V$. Let $A = (R \otimes_{\mathbb{C}} T(V))/\langle W\rangle$ where $T(V)$ denotes the tensor algebra on V.

Definition 6.13 We define the family $\{A(q)\}_{q\in\mathbb{C}'}$ of quadratic $\mathbb{C}$-algebras parametrized by $\mathbb{C}'$ to be $A(q) := R/\mathfrak{m}_q \otimes_R A$ for all $q \in \mathbb{C}'$.

If A is a flat R-module, then the family $\{A(q)\}$ is said to be a flat family, and $A(q)$ is said to be a flat deformation of $A(0)$.

Lemma 6.14 [36, Chapter 4] *If the family $\{A(q)\}_{q\in\mathbb{C}'}$ is flat, then the Hilbert series of $A(q)$ is independent of q for all $q \in \mathbb{C}'$.* ■

For each $q \in \mathbb{C}'$ there is a $\mathbb{C}$-algebra homomorphism $T(V) \twoheadrightarrow A(q)$ given by the composition $T(V) \longrightarrow R \otimes_{\mathbb{C}} T(V) \longrightarrow A \longrightarrow A(q)$, where the first map is $x \mapsto 1 \otimes x$. For all $q \in \mathbb{C}'$, let W_q denote the kernel of the $\mathbb{C}$-algebra homomorphism $T(V)_2 \twoheadrightarrow A(q)_2$, so that $A(q) = T(V)/\langle W_q\rangle$.

Lemma 6.15 [36, Chapter 4] *If A is a flat R-module, then W is a free R-module, and $W_q \cong R/\mathfrak{m}_q \otimes_R W$ for every $q \in \mathbb{C}'$.* ■

Henceforth, we assume that A is a flat R-module and that $A(0)$ is the polynomial ring $S(V)$. In particular, the first condition implies that A is torsion-free (since R is a principal ideal domain) and that Lemmas 6.14 and 6.15 hold. The second condition implies that $A(0) = R \otimes_{\mathbb{C}} S(V)/\langle \mathfrak{m}_0 \rangle$ and that $W_0 = \mathbb{C} \otimes \Lambda^2(V)$ where $\Lambda^2(V)$ denotes the second exterior power of V. Hence, if x, $y \in V$, then there exists $r_{xy} \in W$ of the form

$$r_{xy} = 1 \otimes x \otimes y - 1 \otimes y \otimes x - hb_h(x, y)$$

for some $b_h(x, y) \in R \otimes_{\mathbb{C}} V \otimes_{\mathbb{C}} V$. Since A is torsion-free over R, it follows that $r_{xx} = 0$.

Lemma 6.16 *The module W is generated by the elements r_{xy}.*

Proof. For a contradiction, suppose there exists $w \in W \setminus \sum Rr_{xy}$. Since $A(0) \cong S(V)$, the image of w in W_0 is zero. Therefore, there exists $w' \in R \otimes V \otimes V$ such that $w = hw'$. However, A is torsion-free so $w' \in W$. It follows that $w' \in W \setminus Rr_{xy}$. Repeating this argument, we find $w \in \bigcap(\mathfrak{m}_0{}^n W)$. By Lemma 6.15, W is free, so $\bigcap(\mathfrak{m}_0{}^n W) = (\bigcap \mathfrak{m}_0{}^n)W$. However R is a noetherian domain, so Krull's intersection theorem ensures that $\bigcap \mathfrak{m}_0{}^n = 0$. Hence $w = 0$, which is a contradiction. ■

Let f, $g \in A(0)$, and let $\tilde{f}$, $\tilde{g}$ denote preimages of f, g in A respectively. Then $\tilde{f}\tilde{g} - \tilde{g}\tilde{f} \in Ah$. Since A is torsion-free, we may define a Poisson bracket $\{\ ,\ \}$ on $A(0)$ by [8] to be

$$\{f, g\} = h^{-1}(\tilde{f}\tilde{g} - \tilde{g}\tilde{f}) \quad \text{modulo } Ah,$$

which is independent of the choice of preimages of f and g. In particular, if $x, y \in V$, then $\{x, y\} = b_0(x, y)$.

Since A is graded and since the Poisson bracket is homogeneous of degree zero, the bracket extends to $A(0)[z^{-1}]$, whenever z is a nonzero homogeneous element of $A(0)$, and it then restricts to the degree zero part $(A(0)[z^{-1}])_0$. By covering the projective space $\mathbb{P}(V^*)$ with affine open sets $\{p \in \mathbb{P}(V^*) : z(p) \neq 0\}$, where z is any homogeneous element of $A(0)$, the Poisson bracket on all such $(A(0)[z^{-1}])_0$ induces a Poisson structure on $\mathbb{P}(V^*)$. The maximal connected components of $\mathbb{P}(V^*)$ on which the Poisson structure is nondegenerate are symplectic manifolds, and $\mathbb{P}(V^*)$ is the disjoint union of these symplectic manifolds, which are called *symplectic leaves* (see [15]).

Proposition 6.17 *The projective zero-dimensional symplectic leaves are given by the zero locus of the polynomials* $[x,y,z] := \{x,\ y\}z + \{y,\ z\}x + \{z,\ x\}y$ *for all* $x,\ y,\ z \in A(0)_1$.

Proof. The proof given in [38] for $\mathcal{O}_q(M_n)$ applies. ■

Since A is quadratic, the polynomials $[x,y,z]$ have degree three.

The assumption that A is a flat R-module implies that $W_q \hookrightarrow V \otimes V$. Let $\Gamma_q \subset \mathbb{P}(V^*) \times \mathbb{P}(V^*)$ denote the zero locus of W_q, and let $\pi_i \colon \Gamma_q \to \mathbb{P}(V^*)$ denote the projection onto the ith component for $i = 1, 2$.

Theorem 6.18 *Suppose that for each* $q \in \mathbb{C}'$, Γ_q *is the graph of an automorphism. If* $\pi_1(\Gamma_q)$ *is independent of* q *for all nonzero* $q \in \mathbb{C}'$, *then it is contained in the scheme consisting of the projective zero-dimensional symplectic leaves.*

Proof. Throughout the proof, we assume $q \in \mathbb{C}'$.

The homogeneous coordinate ring of Γ_q is obtained by bilinearizing the defining relations of $A(q)$ as follows. If $x, y \in V$, write the relation $xy - yx - qb_q(x,y)$ as $x_1y_2 - y_1x_2 - qb_q(x,y)_{12}$ which we view as a relation ρ_{xy} for the commutative $\mathbb{C}$-algebra generated by $\{x_1, x_2, y_1, y_2 : x, y \in V\}$. The commutative algebra B_q obtained this way, whose defining relations are the ρ_{xy}, for all $x, y \in V$, is the homogeneous coordinate ring of Γ_q (see [3, §3] and [19]). If $x, y, z \in V$, then, in B_q, we have

$$qx_1b_q(y,z)_{12} = x_1(y_1z_2 - z_1y_2) = q(y_1b_q(x,z)_{12} - z_1b_q(x,y)_{12})$$

from which it follows that if $q \neq 0$, then

$$z_1b_q(x,y)_{12} + x_1b_q(y,z)_{12} + y_1b_q(z,x)_{12} \qquad (*)$$

is zero in B_q. Since Γ_q is the graph of an automorphism σ_q, it follows that $X_2 = \sigma_q(X_1)$ for all $X \in V$. Substitution into $(*)$ yields a polynomial p_q in the variables X_1 where X runs through a basis for V. That is, if $q \neq 0$, then p_q is a relation for the homogeneous coordinate ring of $\pi_1(\Gamma_q)$. However, since, by hypothesis, the $\pi_1(\Gamma_q)$ are equal if $q \neq 0$, it follows that $p_q = f_qP$ where $f_q \in (R/\mathfrak{m}_q) \setminus \{0\}$ and P is a (possibly zero) polynomial in the variables X_1 where X runs through a basis for V.

On the other hand, since σ_0 is the identity, it follows that substituting $q = 0$ and $X_2 = X_1$ for all $X \in V$ into $(*)$ yields

$$z_1 b_0(x_1, y_1) + x_1 b_0(y_1, z_1) + y_1 b_0(z_1, x_1) = p_0 = f_0 P,$$

where $f_0 \in R/\mathfrak{m}_0$. Hence, $p_0 = f_0 f_q^{-1} p_q$ for all $q \neq 0$. However, the polynomials $[x, y, z]$ which define the zero-dimensional symplectic leaves are the polynomials p_0. It follows that all the polynomials $[x, y, z]$, for all $x, y, z \in V$, belong to the ideal defining $\pi_1(\Gamma_q)$ for all $q \neq 0$, which completes the proof. ■

In view of this section and Sections 2-5, the notion of quantum space seems to be a good candidate for the non-commutative space of a quantum group. This is further supported by other examples which have been analyzed in the literature ([18, 19, 37, 39]). Example 6.12, Theorem 6.18 and the striking resemblance between the polynomials defining the projective zero-dimensional symplectic leaves and the quantum Yang-Baxter equation give further credibility to the quantum space of a quantum group being the quantum group's non-commutative space.

Acknowledgment. The author would like to thank S. P. Smith for suggesting that the point scheme of a quadratic quantum group be compared with the scheme of projective zero-dimensional symplectic leaves of the Poisson structure induced on the polynomial ring.

References

[1] M. Artin, Geometry of Quantum Planes, Azumaya Algebras, Actions and Modules, Eds. D. Haile and J. Osterburg, *Contemp. Math.* **124** (1992), 1-15.

[2] M. Artin and W. Schelter, Graded Algebras of Global Dimension 3, *Adv. Math.* **66** (1987), 171-216.

[3] M. Artin, J. Tate and M. Van den Bergh, Some Algebras associated to Automorphisms of Elliptic Curves, *The Grothendieck Festschrift* **1**, 33-85, Eds. P. Cartier et al., Birkhäuser (1990).

[4] M. Artin, J. Tate and M. Van den Bergh, Modules over Regular Algebras of Dimension 3, *Invent. Math.* **106** (1991), 335-388.

[5] M. Artin and M. Van den Bergh, Twisted Homogeneous Coordinate Rings, *J. Alg.* **188** (1990), 249-271.

[6] M. Artin and J. Zhang, Noncommutative Projective Schemes, *Adv. Math.* **109** No. 2 (1994), 228-287.

[7] M. Aubry and J.-M. Lemaire, Zero Divisors in Enveloping Algebras of Graded Lie Algebras, *J. Pure and App. Algebra* **38** (1985), 159-166.

[8] V. G. Drinfel'd, Quantum Groups, *Proc. Int. Cong. Math., Berkeley* **1** (1986), 798-820.

[9] L. D. Faddeev, N. Yu. Reshetikhin and L. A. Takhtadzhyan, Quantization of Lie Groups and Lie Algebras, *Leningrad Math. J.* **1** No. 1 (1990), 193-225.

[10] B. L. Feigin and A. B. Odesskii, Elliptic Sklyanin Algebras, *Func. Anal. Appl.* **23** (1989), 45-54.

[11] M. Jimbo, A q-difference Analogue of $U(\mathfrak{g})$ and the Yang-Baxter Equation, *Lett. Mat. Phys.* **10** (1985), 63-69.

[12] M. Jimbo, Quantum R Matrix for the Generalized Toda System, *Comm. Math. Phys.* **102** (1986), 537-547.

[13] M. Jimbo, A q-analogue of $U(\mathfrak{gl}(n+1))$, Hecke Algebra and the Yang-Baxter Equation, *Lett. Mat. Phys.* **11** (1986), 247-252.

[14] P. Jorgensen and S. P. Smith, Intersection Theory for the Blowup of a Non-commutative surface, in preparation, 1999.

[15] A. A. Kirillov, Local Lie Algebras, *Uspekhi Mat. Nauk* **31** No. 4 (1976), 57-76; English transl. in *Russian Math. Surveys* **31** (1976).

[16] P. P. Kulish and N. Yu. Reshetikhin, The Quantum Linear Problem for the Sine-Gordon Equation and Higher Representations, *Zap. Nauchn. Sem. L.O.M.I.* **101** (1981), 101-110 (Russian).

[17] L. Le Bruyn, Central Singularities of Quantum Spaces, *J. Alg.* **177** (1995), 142-153.

[18] L. Le Bruyn and S. P. Smith, Homogenized $\mathfrak{sl}(2)$, *Proc. Amer. Math. Soc.* **118** (1993), 725-730.

[19] L. Le Bruyn and M. Van den Bergh, On Quantum Spaces of Lie Algebras, *Proc. Amer. Math. Soc.* **119** No. 2 (1993), 407-414.

[20] T. Levasseur, Some Properties of Non-commutative Regular Graded Rings, *Glasgow Math. J.* **34** (1992), 277-300.

[21] T. Levasseur and S. P. Smith, Modules over the 4-Dimensional Sklyanin Algebra, *Bull. Soc. Math. de France* **121** (1993), 35-90.

[22] G. Lusztig, Quantum Deformations of Certain Simple Modules over Enveloping Algebras, *Adv. Math.* **70** (1988), 237-249.

[23] M. Rosso, Representations Irreducibles de Dimension Finie du q-analogue de L'algebre Enveloppante d'une Algebre de Lie Semisimple, *C. R. Acad. Sci. Paris* **305** (1987), 587-590.

[24] M. Rosso, Finite Dimensional Representations of the Quantum Analog of the Enveloping Algebra of a Complex Semisimple Lie algebra, *Comm. Math. Phys.* **117** (1988), 581-593.

[25] J.-P. Serre, Faisceaux Algébriques Cohérents, *Ann. of Math.* **61** (1955), 197-278.

[26] B. Shelton and M. Vancliff, Some Quantum $\mathbb{P}^3$s with One Point, *Comm. Alg.* **27** No. 3 (1999), 1429-1443.

[27] B. Shelton and M. Vancliff, Embedding a Quantum Rank Three Quadric in a Quantum $\mathbb{P}^3$, *Comm. Alg.* **27** No. 6 (1999), 2877-2904.

[28] B. Shelton and M. Vancliff, Schemes of Line Modules, preprint 1999.

[29] E. K. Sklyanin, Some Algebraic Structures connected to the Yang-Baxter Equation, *Func. Anal. Appl.* **16** No. 4 (1982), 27-34.

[30] S. P. Smith, Quantum Groups: An Introduction and Survey for Ring Theorists, *Noncommutative Rings*, Eds. S. Montgomery and L. Small, Springer-Verlag (1992).

[31] S. P. Smith, The 4-dimensional Sklyanin Algebra at Points of Finite Order, preprint 1999.

[32] S. P. Smith and J. T. Stafford, Regularity of the Four Dimensional Sklyanin Algebra, *Compositio Math.* **83** No. 3 (1992), 259-289.

[33] S. P. Smith and J. M. Staniszkis, Irreducible Representations of the 4-Dimensional Sklyanin Algebra at Points of Infinite Order, *J. Alg.* **160** No. 1 (1993), 57-86.

[34] S. P. Smith and J. Zhang, Curves on Quasi-schemes, *Algebras and Representation Theory* **1** No. 4 (1998), 311-351.

[35] T. Tanisaki, Finite Dimensional Representations of Quantum Groups, *Osaka J. Math.* **28** No. 1 (1991), 37-53.

[36] M. Vancliff, The Non-commutative Algebraic Geometry of some Quadratic Algebras, Ph.D. Thesis, University of Washington, August 1993.

[37] M. Vancliff, Quadratic Algebras Associated with the Union of a Quadric and a Line in $\mathbb{P}^3$, *J. Alg.* **165** No. 1 (1994), 63-90.

[38] M. Vancliff, The Defining Relations of Quantum $n \times n$ Matrices, *J. London Math. Soc.* **52** No. 2 (1995), 255-262.

[39] M. Vancliff and K. Van Rompay, Embedding a Quantum Nonsingular Quadric in a Quantum $\mathbb{P}^3$, *J. Alg.* **195** No. 1 (1997), 93-129.

[40] M. Vancliff, K. Van Rompay and L. Willaert, Some Quantum $\mathbb{P}^3$s with Finitely Many Points, *Comm. Alg.* **26** No. 4 (1998), 1193-1208.

[41] M. Van den Bergh, An Example with 20 Points, Notes (1988).

[42] M. Van den Bergh, Blowing up of Non-commutative Smooth Surfaces, *Memoirs Amer. Math. Soc.*, to appear.

[43] M. Van den Bergh, Abstract Blowing Down, *Proc. Amer. Math. Soc.*, **128** No. 2 (2000), 375–381.

[44] K. Van Rompay, Segre Product of Artin-Schelter Regular Algebras of Dimension 2 and Embeddings in Quantum $\mathbb{P}^3$'s, *J. Alg.* **180** No. 2 (1996), 483-512.