Some Quantum $\mathbb{P}^3$s with Finitely Many Points

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Abstract. We consider graded Clifford algebras on n generators in the spirit of Artin, Tate and Van den Bergh's non-commutative algebraic geometry. We give an algorithm for counting the point modules over such an algebra, and prove that a generic graded Clifford algebra on four generators has defining relations which are determined by their zero locus in $\mathbb{P}^3 \times \mathbb{P}^3$. Furthermore, we find a 1-parameter family of iterated Ore extensions on four generators which are deformations of a graded Clifford algebra such that the generic member has precisely one point module and a 1-dimensional family of line modules.

Introduction

A notion of regularity for non-commutative graded algebras was defined in [2], and the regular algebras of global dimension three (generated by degree one elements) were classified in [3, 4]. In so doing, some "non-commutative" geometric techniques were introduced which use certain graded modules (point modules, line modules, etc.) in place of geometric objects (points, lines, etc.). These techniques have subsequently been used in [20, 19] to classify regular algebras of global dimension three which are not generated by degree one elements, and also in [8, 9, 10, 12, 15, 16, 17, 21, 22, 23, 24] to analyse many examples of regular algebras of global dimension four. Nevertheless, to date, relatively little is understood of the

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behaviour of regular algebras of global dimension four and a classification of such algebras is still a long way off.

However, it was pointed out to us by M. Van den Bergh that a generic, quadratic, regular algebra of global dimension four (on four generators with six defining relations) has a 1-dimensional family of line modules and at most twenty point modules. A family of examples was found by Stafford where each algebra has a 1-dimensional family of line modules but infinitely many point modules (see [17, 18]). On the other hand, a class of examples (graded Clifford algebras) was introduced by Van den Bergh in [25] where the generic member has precisely twenty point modules and a 2-dimensional family of line modules (but no explicit example was given). Our main objective in this paper is to deform a certain Clifford algebra to produce a family of regular algebras of global dimension four such that the generic member has precisely one point module and a 1-dimensional family of line modules.

The paper is organized as follows. The purpose of §1 is to give an algorithm in Theorem 1.7 for counting the point modules over Clifford algebras (on n generators). In §2 we give an explicit example of a regular Clifford algebra on four generators which has precisely twenty point modules. We prove in Proposition 2.2 that a generic Clifford algebra on four generators has defining relations which are determined by their zero locus in $\mathbb{P}^3 \times \mathbb{P}^3$.

In §3 we focus on a family of iterated Ore extensions which are deformations of a regular Clifford algebra on four generators. The Clifford algebra and the generic deformation have precisely one point module. We prove in Proposition 3.5 that the generic deformation has a 1-dimensional family of line modules.

The examples in §§2,3 have line modules which do not map onto any point module. These are the first examples known to have such line modules. At the end of §3 we pose some related questions which are still open.

1. Graded Clifford Algebras

Throughout this section, k denotes an algebraically closed field of characteristic different from two. Let $R = k[y_1, \dots, y_n]$ denote the commutative polynomial ring on n variables and let $Y = (Y_{ij}) \in M_n(R)$ denote a symmetric matrix whose entries are homogeneous linear polynomials in the variables y_i.

Definition 1.1. In the terminology of [25] and [7, §4], the *Clifford algebra* $A = A(Y)$ over R associated to Y is the k-algebra on generators $x_1, \dots, x_n$, $y_1, \dots, y_n$ with defining relations $x_i x_j + x_j x_i = Y_{ij}$ for all i, j, and y_i central for all i.

We define a grading on A by declaring $\deg(x_i) = 1$ and $\deg(y_i) = 2$ for all i.

Writing $Y = Y_1 y_1 + \cdots + Y_n y_n$ where the $Y_i \in M_n(k)$ are symmetric matrices, we may associate to the matrix Y an n-dimensional linear system $\mathcal{Q} = kQ_1 + \cdots + kQ_n$ of quadrics $Q_1, \ldots, Q_n \subset \mathbb{P}^{n-1}$ by taking each Q_i to be the quadric in $\mathbb{P}^{n-1}$ corresponding to Y_i; that is, $Q_i = \{y \in \mathbb{P}^{n-1} : y^T Y_i y = 0\}$. A base point of $\mathcal{Q}$ is a common point of intersection of all the Q_i.

The system $\mathcal{Q}$ determines certain homological properties of $A(Y)$, such as regularity, as follows.

Definition 1.2.

(1) [2, Page 171] A connected positively graded k-algebra B is called regular if
   (a) the global (homological) dimension of B (gldim(B)) is finite,
   (b) the Gelfand-Kirillov dimension of B (GKdim(B)) is finite, and
   (c) B is Gorenstein; that is, $\mathrm{Ext}^q_B(k, B) = \delta^q_n k$ where $n = \mathrm{gldim}(B)$.

(2) [11, §2] A connected positively graded k-algebra B is called Auslander-regular if
   (a) gldim(B) is finite, and
   (b) for every finitely generated B-module M, for every $i \geq 0$ and for every B-submodule N of $\mathrm{Ext}^i_B(M, B)$, we have $j(N) \geq i$, where

$$j(N) = \inf\{j : \mathrm{Ext}^j_B(N, B) \neq 0\}.$$

(3) [11, §5] A connected positively graded k-algebra B is said to satisfy the Cohen-Macaulay property if $\mathrm{GKdim}(M) + j(M) = \mathrm{GKdim}(B)$ for all nonzero finitely generated B-modules M.

Proposition 1.3. [7, Proposition 7] *In the above notation, the Clifford algebra A is a quadratic Auslander-regular algebra of global dimension n (which satisfies the Cohen-Macaulay property) if and only if the system $\mathcal{Q}$ has no base points.* ∎

Henceforth we assume that $\mathcal{Q}$ has no base points. In particular, A is generated by the x_i only, and so is a finite module over R and hence Noetherian. By [5, §3], A is the enveloping algebra of a Lie superalgebra and so has $n(n-1)/2$ defining relations and they are of the form $\sum_{ij} \alpha_{ijm}(x_i x_j + x_j x_i) = 0$ where the $\alpha_{ijm} \in k$, and A has Hilbert series $H_A(t) = (1-t)^{-n}$, which is the same as that of the polynomial ring on n variables. It therefore follows from Proposition 1.3 and [11, Theorem 6.3] that A is regular, and so, in the language of [1], Proj(A) may be viewed as a quantum $\mathbb{P}^{n-1}$.

Definition 1.4. [3, 3.8] Let B denote a positively graded k-algebra which is generated by its degree one elements. A (left) point module (respectively, line module) over B is a cyclic graded B-module $M = \oplus_{i \geq 0} M_i$ such that $M = BM_0$ and $H_M(t) = (1-t)^{-1}$ (respectively, $H_M(t) = (1-t)^{-2}$). Point modules give (some) points of $\mathrm{Proj}(B)$, whereas line modules give (some) lines.

We write $A = T(V)/\langle W \rangle$ where $V = kx_1 \oplus \cdots \oplus kx_n$, $T(V)$ denotes the tensor algebra on V and $W \subset V \otimes V$ is the span of the defining relations of A. Let $\Gamma = \{(a,b) \in \mathbb{P}^{n-1} \times \mathbb{P}^{n-1} : f(a,b) = 0 \text{ for all } f \in W\}$, and let $\pi_i : \Gamma \to \mathbb{P}^{n-1}$ denote the i'th projection map. Since $W = \text{ Span } \{\sum_{ij} \alpha_{ijm}(x_i \otimes x_j + x_j \otimes x_i) : 1 \leq m \leq n(n-1)/2\}$, it follows that $\pi_1(\Gamma) = \pi_2(\Gamma)$, and so by [3, §3], the isomorphism classes of left (respectively, right) point modules over A are parametrized by $\pi_i(\Gamma)$. Owing to the symmetry of the defining relations, if $(a,b) \in \Gamma$, then so is (b,a); in other words, Γ is the graph of an automorphism $\sigma : \pi_1(\Gamma) \to \pi_1(\Gamma)$ where σ has order two.

In this situation, a point $(a,b) \in \Gamma$ if and only if $\sum_{ij} \alpha_{ijm}(a_i b_j + a_j b_i) = 0$ for all $m \leq n(n-1)/2$. That is, $(a,b) \in \Gamma$ if and only if the symmetric matrix $ab^T + ba^T = (a_i b_j + a_j b_i)$ is a zero of $\sum_{ij} \alpha_{ijm} X_{ij}$ for all $m \leq n(n-1)/2$, where X_{ij} denotes the ij'th coordinate function. In view of the discussion in [25] for the $n = 4$ case, we consider symmetric $n \times n$ matrices as $\mathbb{P}^N$, where $N = (n/2)(n+1) - 1$, and define a map $\phi : \mathbb{P}^{n-1} \times \mathbb{P}^{n-1} \to \mathbb{P}^N$ by $\phi(u,v) = uv^T + vu^T$ (which is a matrix of rank ≤ 2). Since k is algebraically closed of characteristic other than two and since $\mathrm{GL}_n(k)$ acts transitively on matrices of the same rank, the image of ϕ consists of all symmetric $n \times n$ matrices X such that $\mathrm{rank}(X) \leq 2$.

Proposition 1.5. *If $\mathcal{Q}$ has no base points, then the image of $\phi|_\Gamma$ consists of those matrices $X \in \mathbb{P}(\sum_{i=1}^n kY_i)$ such that rank$(X) \leq 2$.*

Proof. As above, we write the defining relations of A as $\sum_{ij} \alpha_{ijm}(x_i x_j + x_j x_i) = 0$ and let X_{ij} denote the ij'th coordinate function on $\mathbb{P}^N$. If $(a_{ij}) = \sum_{t=1}^n a_t Y_t$ where the $a_t \in k$, then $(a_{ij}) = Y|_{(a_1,\dots,a_n)}$ and $(a_{ij}) \in \mathcal{V}(\sum_{ij} \alpha_{ijm} X_{ij})$ for all $m \leq n(n-1)/2$. Conversely, if $(a_{ij}) \in \mathcal{V}(\sum_{ij} \alpha_{ijm} X_{ij})$ for all $m \leq n(n-1)/2$, then for $1 \leq t \leq n$, we may define $\beta_{ijt} \in k$ from the formula $y_t = \sum_{ij} \beta_{ijt} Y_{ij}$ (since $\mathcal{Q}$ has no base points), and set $a_t = \sum_{ij} \beta_{ijt} a_{ij}$; this gives $(a_{ij}) \in k(\sum_{t=1}^n a_t Y_t)$. The result follows since the image of ϕ equals $\{X \in \mathbb{P}^N : \mathrm{rank}(X) \leq 2\}$. ■

The reader might wish to compare Proposition 1.5 with [7, Proposition 9 and Remark 2]. Proposition 1.5 combined with the following result provides an algorithm for counting the number of point modules over A. Let $\Delta = \{(u,u) \in \mathbb{P}^{n-1} \times \mathbb{P}^{n-1}\}$.

Lemma 1.6. [25] *The restriction of* ϕ *to* $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1} \setminus \Delta$ *has degree two and is unramified, whereas* $\phi|_\Delta$ *is one-to-one.*

Proof. Since $\mathrm{GL}_n(k)$ acts transitively on matrices of the same rank, it suffices to check the result for a rank one matrix and a rank two matrix, such as $X = \mathrm{diag}(2, \lambda, 0, \dots, 0)$ for any $\lambda \in k$. However, $X = \phi(u, v)$ where u, v are uniquely defined by $u = (1, \pm i\sqrt{\lambda/2}, 0, \dots, 0)$ and $v = (1, \mp i\sqrt{\lambda/2}, 0, \dots, 0)$. ■

Theorem 1.7. *If* $\mathcal{Q}$ *has no base points, then the number of isomorphism classes of left (respectively, right) point modules over* A *is equal to* $2r_2 + r_1 \in \mathbb{N} \cup \{0, \infty\}$ *where* r_j *denotes the number of matrices in* $\mathbb{P}(\sum_{i=1}^n kY_i)$ *which have rank* j*. If the number of left (respectively, right) point modules is finite, then* $r_1 \in \{0, 1\}$*.*

Proof. The first statement follows from Proposition 1.5 and Lemma 1.6. To prove the second statement, suppose there exist two linearly independent rank one matrices $X_1, X_2 \in \mathbb{P}(\sum_{i=1}^n kY_i)$. Since each X_i is symmetric, we may assume that $\mathrm{Ker}(X_1) \neq \mathrm{Ker}(X_2)$. It follows that $\dim(\mathrm{Ker}(X_1) \cap \mathrm{Ker}(X_2)) = n - 2$, so that every matrix in $k^\times X_1 + k^\times X_2$ has rank two, which implies that A has infinitely many point modules. ■

2. Algebras with Twenty Point Modules

Throughout this section we continue to focus on graded Clifford algebras with the notation as in §1, but with the assumption that $n = 4$. Let k denote an algebraically closed field of characteristic different from two. If the symmetric matrix Y is generic, then Van den Bergh proves in [25] that the Clifford algebra $A(Y)$ has precisely twenty point modules. Since [25] is unpublished, we give the proof below in Theorem 2.1. However, no explicit example is known of such an algebra, so we devote most of this section to an example which has exactly twenty point modules. We also show that if $\mathrm{char}(k) = 0$ and Y is generic, then the defining relations of $A(Y)$ are determined by the geometric data.

Theorem 2.1. [25] *If the symmetric matrix* Y *is generic, then the Clifford algebra* $A(Y)$ *has exactly twenty point modules.*

Proof. As in §1, identify $\mathbb{P}^9$ with symmetric 4×4 matrices. Let $\mathcal{V}$ denote the subvariety of $\mathbb{P}^9$ consisting of the matrices of rank ≤ 2; $\mathcal{V}$ has dimension six and degree ten. If Y is generic, then $\mathcal{Q}$ has no base points and $\mathbb{P}(\sum_{i=1}^n kY_i)$ is a generic $\mathbb{P}^3$ in $\mathbb{P}^9$. Hence, by Proposition 1.5, the image of $\phi|_\Gamma$ is equal to the intersection of $\mathcal{V}$ with a generic $\mathbb{P}^3$ in $\mathbb{P}^9$.

By counting dimensions (namely, $\dim(\mathcal{V})+\dim(\mathbb{P}^3)-\dim(\mathbb{P}^9)=0$), we have that the intersection of $\mathcal{V}$ with a generic $\mathbb{P}^3$ in $\mathbb{P}^9$ consists of at most finitely many points. Since $\mathcal{V}$ has degree ten, Bertini's theorem implies that such an intersection consists of exactly ten points. Moreover, the variety consisting of rank one matrices has dimension strictly smaller then $\dim(\mathcal{V})$, and so has empty intersection with a generic $\mathbb{P}^3$ in $\mathbb{P}^9$. The result follows by applying Lemma 1.6 (or Theorem 1.7). ■

Henceforth, we assume that k is an algebraically closed field of characteristic zero.

Proposition 2.2. *If the symmetric matrix Y is generic, then the defining relations of the Clifford algebra $A(Y)$ are determined by the geometric data; that is,*

$$A(Y)=\frac{T(V)}{\langle f\in V\otimes V : f(\Gamma)=0\rangle},$$

where Γ is the zero locus in $\mathbb{P}^3\times\mathbb{P}^3$ of the defining relations of $A(Y)$.

Proof. Suppose that Y is generic and, as in §1, let W denote the span of the defining relations of A. Let $f\in V\otimes V\setminus W$ and suppose $f(\Gamma)=0$. Since $\dim(W)=6$, we may write f as a linear combination of ten basis vectors from $V\otimes V$, with coefficients $\alpha_1,\ldots,\alpha_{10}\in k$. By Theorem 2.1, evaluating f on Γ yields twenty homogeneous linear equations in the α_i. One would expect this system of equations to have rank ten, so that the only solution would be $\alpha_i=0$ for all i (in which case the result would follow), but it is conceivable that some general property of Clifford algebras (such as the symmetry in the relations) or the structure of Γ forces the system to have rank <10. However, if one can find a (possibly nongeneric) regular Clifford algebra which has twenty point modules for which the proposition holds (i.e., the associated system of twenty homogeneous linear equations has rank ten), then it would follow that the same is true for a generic Clifford algebra. We construct such an example below to complete the proof. ■

Let $A=k[x_1,\ldots,x_4]$ with defining relations

$$\begin{aligned} x_1x_2+x_2x_1&=x_3^2, & x_2x_3+x_3x_2&=x_1^2,\\ x_1x_3+x_3x_1&=x_4^2, & x_3x_4+x_4x_3&=x_1^2,\\ x_1x_4+x_4x_1&=x_2^2, & x_2x_4+x_4x_2&=x_3^2+x_4^2, \end{aligned}$$

so that A is the Clifford algebra $A(Y)$ where

$$Y=\begin{bmatrix} 2x_1^2 & x_3^2 & x_4^2 & x_2^2\\ x_3^2 & 2x_2^2 & x_1^2 & x_3^2+x_4^2\\ x_4^2 & x_1^2 & 2x_3^2 & x_1^2\\ x_2^2 & x_3^2+x_4^2 & x_1^2 & 2x_4^2 \end{bmatrix},$$

and $y_i = x_i^2$ for all i. The linear system $\mathcal{Q}$ of quadrics associated to Y is given by the matrices

$$Y_1 = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \qquad Y_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

$$Y_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \qquad Y_4 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix},$$

which correspond to the quadratic forms

$$\begin{array}{ll} x_1^2 + x_2x_3 + x_3x_4, & x_1x_4 + x_2^2, \\ x_1x_2 + x_2x_4 + x_3^2, & x_1x_3 + x_2x_4 + x_4^2. \end{array}$$

This system has no base points so, by Proposition 1.3, the algebra A is Auslander-regular of global dimension four and satisfies the Cohen-Macaulay property. We first show that A has twenty point modules.

By Theorem 1.7, we seek those $X = \sum_{i=1}^{n} \alpha_i Y_i$ where $\alpha_i \in k$ such that $\operatorname{rank}(X) \le 2$; that is,

$$X = \begin{bmatrix} 2\alpha_1 & \alpha_3 & \alpha_4 & \alpha_2 \\ \alpha_3 & 2\alpha_2 & \alpha_1 & \alpha_3 + \alpha_4 \\ \alpha_4 & \alpha_1 & 2\alpha_3 & \alpha_1 \\ \alpha_2 & \alpha_3 + \alpha_4 & \alpha_1 & 2\alpha_4 \end{bmatrix}$$

and $\operatorname{rank}(X) \le 2$. Solving for the zeros of the 3×3 minors of X corresponding to the 11, 12 and 22 entries of X yields three rank two solutions for X corresponding to: $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (2, 1, 1, 1)$, $(-1 \pm \sqrt{2}, 1, 1, 1)$, together with the solutions corresponding to the α_3 which are zeros of the polynomial $p(x) = 1 + 4x + 6x^2 - 70x^3 - 96x^4 - 24x^5 + 77x^6 + 29x^7$ — but it is conceivable that the latter solutions do not correspond to matrices of rank ≤ 2. By approximating the zeros of p with *Mathematica*, one finds that the corresponding solutions for X have rank strictly bigger than one and no row is zero. It follows that the other 3×3 minors of these solutions are zero and so these solutions have rank two. By Theorem 1.7, since all the solutions for X are distinct, A has exactly twenty point modules.

The computation to verify that the defining relations of this example are determined by the point modules is left to the reader. (An outline of the computation may be obtained by contacting the first author.)

Remarks 2.3.

(a) In §3, a certain family of regular algebras having a unique point module is considered. In [14, §1], it is proved that the defining relations of any member of the family are determined by the scheme parametrizing the point module; the scheme consists of the point as a subscheme of $\mathbb{P}^3 \times \mathbb{P}^3$ and its multiplicity (which is twenty). This result is surprising given that there is only one point module; the proof is almost identical to the proof of Proposition 2.2, except that the analysis is carried out in the (20-dimensional) stalk of the structure sheaf of the scheme.

(b) The example with twenty point modules in this section has line modules which do not map onto any point module. For instance, by [12, Proposition 2.8], the line $\ell = \mathcal{V}(x_3, x_2 - x_4)$ corresponds to a line module $M(\ell)$ since $(x_2 - x_4)x_3 + x_3(x_2 - x_4) = 0$, but $M(\ell)$ does not map onto any point module since none of the twenty points lie in the plane $\mathcal{V}(x_3)$.

3. Deformations with One Point Module

Until Remarks 3.6, we assume that k is an algebraically closed field of characteristic zero.

By Theorem 2.1, if $n = 4$ and if $\mathcal{Q}$ has no base points, then a Clifford algebra which has finitely many point modules has at most twenty point modules, and in the previous section we gave an example with twenty point modules. On the other hand, the proof of Theorem 2.1 shows that if $n = 4$ and if $\mathcal{Q}$ has no base points, then every Clifford algebra has at least one point module. In this section we present a family of iterated Ore extensions which are deformations of a Clifford algebra. Both the Clifford algebra and the generic deformation have precisely one point module, but the generic deformation has only a 1-dimensional family of line modules, which is the minimal possible dimension of the variety of line modules for a regular algebra of global dimension four.

Definition 3.1. Let $A(q, \lambda) = k[x_1, \ldots, x_4]$ with defining relations

$$\begin{aligned} x_1x_2 - qx_2x_1 &= \lambda x_4^2, & x_2x_3 &= qx_3x_2, \\ x_1x_3 - qx_3x_1 &= x_2^2, & x_3x_4 &= qx_4x_3, \\ x_1x_4 - qx_4x_1 &= x_3^2, & x_4x_2 &= qx_2x_4, \end{aligned}$$

where $q, \lambda \in k^\times$ and $q^4 = 1$.

The specialization $A(-1,1)$ is a Clifford algebra $A(Y)$ where

$$Y = \begin{bmatrix} 2x_1^2 & x_4^2 & x_2^2 & x_3^2 \\ x_4^2 & 2x_2^2 & 0 & 0 \\ x_2^2 & 0 & 2x_3^2 & 0 \\ x_3^2 & 0 & 0 & 2x_4^2 \end{bmatrix},$$

and $y_i = x_i^2$ for all i.

Lemma 3.2. *For every $\lambda \in k^\times$, $A(q,\lambda)$ is an iterated Ore extension of the field k.*

Note . If the assumption "$q^4 = 1$" is removed, then this result is false, and, in fact, the corresponding algebra is not regular. For more details, see [14, §1].

Proof. It is straightforward to see that the subalgebra $B(q)$ of $A(q,\lambda)$ generated by x_2, x_3 and x_4 is an iterated Ore extension of k. On $B(q)$ we define an automorphism ψ by the formula $\psi(x_i) = qx_i$ for all $i = 2,3,4$ and a ψ-derivation δ by

$$\delta(x_2) = \lambda x_4^2, \qquad \delta(x_3) = x_2^2, \qquad \delta(x_4) = x_3^2;$$

which we verify is a ψ-derivation of $B(q)$ as follows. We have

$$\begin{aligned} \delta(x_2x_3 - qx_3x_2) &= \psi(x_2)\delta(x_3) + \delta(x_2)x_3 - q(\psi(x_3)\delta(x_2) + \delta(x_3)x_2) \\ &= qx_2^3 + \lambda x_4^2 x_3 - q^2\lambda x_3 x_4^2 - qx_2^3, \end{aligned}$$

and the last expression belongs to the ideal generated by the defining relations of $B(q)$ since $q^4 = 1$. Similarly, we have

$$\delta(x_3x_4 - qx_4x_3) = qx_3^3 + x_2^2x_4 - q^2x_4x_2^2 - qx_3^3$$

and

$$\delta(x_4x_2 - qx_2x_4) = q\lambda x_4^3 + x_3^2x_2 - q^2x_2x_3^2 - q\lambda x_4^3,$$

which both belong to the ideal generated by the defining relations of $B(q)$ since $q^4 = 1$. It follows that $A(q,\lambda) = B(q)[x_1; \psi, \delta]$. ■

It follows from Lemma 3.2, [6, Theorem 4.2] and [13, Page 184] that $A(q,\lambda)$ is Auslander-regular of global dimension four, satisfies the Cohen-Macaulay property and is Noetherian with Hilbert series $H(t) = (1-t)^{-4}$. (Moreover, one can show that $A(q,\lambda)$ has many homogeneous, central elements of degree four (since $q^4 = 1$), and that $A(q,\lambda)$ is a finite module over its centre.)

It is straightforward to see by computation that if $q = 1$, then $A(q,\lambda)$ has infinitely many point modules. Our interest is in those $A(q,\lambda)$ such that $q \neq 1$, since, in this case, $A(q,\lambda)$ has a unique point module, as the following lemma demonstrates.

Lemma 3.3. *If $q \neq 1$, then $A(q, \lambda)$ has precisely one point module; namely, the point module corresponding to the point* $((1,0,0,0),(1,0,0,0)) \in \mathbb{P}^3 \times \mathbb{P}^3$.

Proof. Let $\Gamma \subset \mathbb{P}^3 \times \mathbb{P}^3$ denote the zero locus of the defining relations of $A(q, \lambda)$, and let $(a, b) = ((a_i), (b_i)) \in \Gamma$. If $q \neq 1$, then the three relations between x_2, x_3, x_4 imply that (a, b) is either $((a_1, 0, a_3, a_4),\ (b_1, 0, a_3, qa_4))$ or $((a_1, a_2, 0, a_4),\ (b_1, qa_2, 0, a_4))$ or $((a_1, a_2, a_3, 0),$ $(b_1, a_2, qa_3, 0))$, for some $b_1 \in k$. The remaining three relations force $a_2 = a_3 = a_4 = 0$, so that $\Gamma = \{((1,0,0,0),(1,0,0,0))\}$. The result now follows from [3, §3]. ■

Remark 3.4. It follows that if $q \neq 1$, then the defining relations of $A(q, \lambda)$ are not determined by the point variety of $A(q, \lambda)$; namely, the variety corresponding to the unique point module together with a certain automorphism of that variety. This observation contrasts with the 20-point example in §2. However, it is proved in [14, §1] that if $q \neq 1$, then $A(q, \lambda)$ is determined by its point scheme; meaning, the scheme parametrizing the point module together with a certain automorphism of the scheme.

By [12, Proposition 2.8], the line modules over $A(q, \lambda)$ are given by the rank two 2-tensors in the span of the six defining relations. Any 2-tensor corresponds to a 4×4 matrix of the same rank. Consequently, the dimension of the variety of line modules of a generic, quadratic, regular algebra of global dimension four (with six defining relations) is $11 + 5 - 15 = 1$, where $15 = \dim(\mathbb{P}(4 \times 4 \text{ matrices}))$, $5 = \dim(\mathbb{P}(\text{span of defining relations}))$ and $11 = \dim(\mathbb{P}(4 \times 4 \text{ rank} \leq 2 \text{ matrices}))$. However, in the case of Clifford algebras, since the defining relations are symmetric, the same computation may be done with $\mathbb{P}^9$ giving $6 + 5 - 9 = 2$. (We thank M. Van den Bergh for pointing out to us these dimension arguments.)

It follows that the Clifford algebra $A(-1, 1)$ has at least a 2-dimensional family of line modules. In fact, $A(-1, 1)$ has a 1-dimensional family of line modules whose corresponding lines in $\mathbb{P}^3$ pass through $p = (1,0,0,0)$, but it also has a 2-dimensional family of line modules whose corresponding lines do not pass through p.

Proposition 3.5. *If $q^2 = -1$, then the variety of line modules of $A(q, \lambda)$ is 1-dimensional for all $\lambda \in k^\times$.*

Proof. As mentioned above, associate to each defining relation $\sum_{ij} \beta_{ij}^{(m)} x_i x_j = 0$ of $A(q, \lambda)$ the matrix $(\beta_{ij}^{(m)}) = Z_m$ for $m = 1, \ldots, 6$. We need to find all matrices $Z \in \mathbb{P}(\sum_{m=1}^{6} kZ_m)$

which have rank two; that is, we seek all $(z_1, \dots, z_6) \in \mathbb{P}^5$ such that the matrix

$$Z = \begin{bmatrix} 0 & z_4 & z_5 & z_6 \\ -qz_4 & -z_5 & z_1 & -qz_3 \\ -qz_5 & -qz_1 & -z_6 & z_2 \\ -qz_6 & z_3 & -qz_2 & -\lambda z_4 \end{bmatrix} \qquad (*)$$

has rank two, where $q^2 = -1$.

Suppose Z is a rank two matrix of the form $(*)$. If $z_6 = 0$, then $z_4 = z_5 = z_1 z_2 z_3 = 0$ (since $q \neq 1$), which gives three 1-dimensional components. If instead $z_6 = 1$, then z_1, z_2, z_5 may be found easily in terms of z_3 and z_4. Hence, the component lying in $\mathbb{P}^5 \setminus \mathcal{V}(z_6)$ is given by the equation

$$1 + z_3 z_4 (q-1)[(q-1)z_3 - \lambda z_4^2]^2 = 0,$$

and so is also 1-dimensional. ■

The line in $\mathbb{P}^3$ which corresponds to the left (respectively, right) line module corresponding to Z in the proof of Proposition 3.5 is given by $\{x \in \mathbb{P}^3 : Zx = 0\}$ (respectively, $\{x \in \mathbb{P}^3 : x^T Z = 0\}$). The lines in $\mathbb{P}^3$ corresponding to the (left or right) line modules given by $z_6 = 0$ in the proof of Proposition 3.5 all pass through $p = (1, 0, 0, 0)$, but those corresponding to $z_6 = 1$ do not pass through p.

We close the paper with some open questions.

Remarks 3.6.

(1) Is it possible to deform the example of §2 in such a way that the deformation is still regular of global dimension four with exactly twenty point modules but is not a Clifford algebra? If this were possible, then such an algebra would probably have a 1-dimensional family of line modules, and so would be a candidate for a generic regular algebra of global dimension four.

(2) By §§2,3, there exist examples of Clifford algebras with twenty point modules and one point module (an example is also known with fourteen point modules). However it is still unknown whether or not there exist regular Clifford algebras on four generators with m point modules for all $m \in \{1, \dots, 20\}$.

(3) The examples in §§2,3 have line modules which do not map onto any point module. However the following is still unknown: if A is a quadratic regular algebra of global dimension four, and if M is a point module over A, does there exist a line module over A which maps onto M?

(4) To date every example known of a regular algebra of global dimension four which has a finite number of point modules is a finite module over its centre. Is this true for all such algebras?

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