SCHEMES OF LINE MODULES I

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Abstract. We prove that there exists a scheme which represents the functor of line modules over a graded algebra, and call it the *line scheme* of the algebra. We study its properties and its relationship to the point scheme. If the line scheme of a quadratic, Auslander-regular algebra of global dimension four has dimension one, then it determines the defining relations of the algebra.

Moreover, we prove the following counter-intuitive result: if the zero locus of the defining relations of a quadratic (not necessarily regular) algebra on four generators with six defining relations is finite, then it determines the defining relations of the algebra. Although this result is non-commutative in nature, its proof uses only commutative theory.

We also use the structure of the line scheme and the point scheme of a four-dimensional regular algebra to determine basic incidence relations between line modules and point modules.

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Introduction

Non-commutative algebraic geometry was introduced in the late 1980s by M. Artin, J. Tate and M. Van den Bergh as a tool for classifying and understanding certain non-commutative algebras. Roughly speaking, Artin, Tate and Van den Bergh used the category of graded modules over a non-commutative algebra as the space in which to do geometry; the geometric objects being certain graded modules, *linear modules*, which play the role of linear objects (points, lines, and so forth).

This geometric theory has been most successful in analysing algebras which are "deformations" in some sense of polynomial rings. One such class of algebras are Artin-Schelter regular algebras which are graded algebras with the same good growth and homological properties as polynomial rings ([2]). The theory is also applicable to other types of algebras ([27, Chapter 2]). An attractive feature of the theory is that it recovers commutative algebro-geometric results in addition to producing new results for non-commutative algebras.

Throughout the article, k denotes an algebraically closed field such that $\text{char}(k) \neq 2$. Let B denote a noetherian, positively graded, connected k-algebra generated by homogeneous elements of degree one. We write gr-B for the category of finitely generated, graded B-modules.

Definition. [1] Define $\text{Proj}\, B$ to be the triple $(\,(\text{gr-B})/\mathcal{T}, \mathcal{O}, \sigma)$ where $\mathcal{T}$ denotes the subcategory of gr-B of torsion modules, $\mathcal{O}$ denotes an object of $(\text{gr-}B)/\mathcal{T}$ which is represented by the right module B, and σ is the operation $\mathcal{M} \to \mathcal{M}[1]$ on $(\text{gr-}B)/\mathcal{T}$ induced by the shift of degree on a B-module. A quantum $\mathbb{P}^2$, or quantum projective plane, is $\text{Proj}\, B$ where B is a quadratic Artin-Schelter regular algebra of global dimension three.

The role of $\text{Proj}\, B$ is similar to that of the classical notion of Proj, in that it should be viewed as a geometric space which depicts properties of B. For instance, the finite-dimensional simple modules of the Sklyanin algebra on four generators are quotients of line modules ([22]), so, in that case, the lines of $\text{Proj}\, B$ are tools for classifying the finite-dimensional simple modules. In general, a quantum space should be Proj B for some B, but it is unclear if regularity of B alone is sufficient to guarantee that B may be sensibly viewed as a deformation of a polynomial ring. As such, there is still debate over the definition of a quantum $\mathbb{P}^3$ and of a quantum projective space.

In [3], it was shown that if a graded algebra A is connected and generated by degree-one elements, then there is a (commutative) scheme, named *the point scheme* in [31], which represents the functor of flat families of point modules of A. For 3-dimensional Artin-Schelter regular algebras, the point scheme is isomorphic to the zero locus of the defining relations of the algebra, and is typically the graph of an automorphism σ of a cubic divisor in $\mathbb{P}^2$. In this situation, the automorphism σ encodes the multiplication of the algebra in such a way that the algebra is a finite module over its centre if and only if σ has finite order.

The Artin-Schelter regular algebras of global dimension three were classified in [2, 3, 4, 25, 26], and the main tool used in the classification was the point scheme. Classification of the Artin-Schelter regular algebras of global dimension four is still an unsolved problem, in spite of the many examples which have been analysed, such as many quantum groups, graded Clifford algebras, homogenizations of Lie algebras, deformations of some Poisson algebras and Sklyanin algebras ([7, 8, 9, 11, 13, 16, 20, 21, 22, 23, 24, 28, 29, 30, 33, 34]). A class of algebras, called Auslander-regular algebras, are Artin-Schelter regular if they have polynomial growth ([12]). The studied examples are all Auslander-regular algebras, but such algebras of global dimension four have also not been classified. One difficulty is that, in contrast to the lower-dimensional cases, the point scheme need not determine the defining relations of the algebra, even if the algebra is quadratic. Our main objective in this paper is to overcome this difficulty by introducing a higher-dimensional version of the point scheme, namely a notion of *line scheme*. We also seek methods which compute the line scheme easily.

In §1, we prove that there is a scheme which represents the functor of flat families of line modules of a graded algebra A, and we call this scheme the *line scheme* of A.

Additional hypotheses are imposed in §2 in order to describe different incarnations of the line scheme so that it may be computed for quadratic, Auslander-regular algebras of global dimension four. The hypotheses are Conditions 2.1-2.3 and Condition 2.8; they ensure that the graded algebra A is a domain, that A has the same Hilbert series as that of the polynomial ring on four variables and that certain restrictions are imposed on line modules and plane modules over A. In particular, although the line scheme may be described as a subscheme of a

Grassmannian scheme, in some cases it is best described as a subscheme of the projectivization of the defining relations of A.

In §3, we assume that A is a quadratic, noetherian, Auslander-regular algebra of global dimension four which satisfies the Cohen-Macaulay property. Such an algebra satisfies Conditions 2.1-2.3 and Condition 2.8, and there is an interesting interplay between the point modules and the line modules. In particular, every point module is a quotient of some line module, and if a point module is covered by at most finitely many line modules, then it is covered by six line modules (counting multiplicity). Moreover, if a line module maps onto four nonisomorphic point modules, then every point on the corresponding line corresponds to a point module.

One of our main results is proved in §4 as Theorem 4.1 and is the following.

> Let A denote a quadratic algebra on four generators with six defining relations, and let $\Gamma_2 \subset \mathbb{P}(A_1^*) \times \mathbb{P}(A_1^*)$ denote the zero locus of the defining relations of A. If Γ_2 is finite, then the span of the defining relations of A consists of all those degree-two forms which vanish on Γ_2.

Although this result is non-commutative in nature, its proof uses only commutative algebra. An analogous result is proved in Theorem 4.3 for quadratic, Auslander-regular algebras satisfying the hypotheses of §3; that is, for such an algebra, if the line scheme has dimension one, then it determines the defining relations of the algebra.

In a sequel to this paper, [18], we will give a simple way to find explicit coordinate descriptions of line schemes in terms of Plücker coordinates. We will then consider several classes of examples of quadratic, Auslander-regular algebras of global dimension four, such as Clifford algebras ([7, 16]), quadric and quadric-line algebras ([17, 31, 32]), and homogenized $\mathfrak{sl}(2)$ ([9]). In addition, we will give general criteria for the line scheme of the algebra to be at least 2-dimensional.

1. Schemes of Linear Modules for Graded Algebras

Let k denote an algebraically closed field such that $\mathrm{char}(k) \neq 2$ and let v be a fixed positive integer. Let $T = T(k)$ be the free k-algebra on generators $x_1, \ldots, x_v$, where T is graded via $\deg(x_i) = 1$ for all $i \in \{1, \ldots, v\}$. Let I be a fixed homogeneous ideal in T and let $A = T/I$. We will take the generators x_i to be a basis for the dual, V^*, of a fixed v-dimensional vector space $V = k^v$, and we identify T with the graded tensor algebra $T(V^*) = \bigoplus_n (V^*)^{\otimes n}$.

For any commutative k-algebra R we denote the graded algebra $A \otimes R$ by $A(R)$. This algebra is graded by the Kunneth formula where we take R to be graded by $R = R_0$. Unless otherwise stated, all $A(R)$-modules will be right modules and all homomorphisms will be graded, degree-zero homomorphisms.

Let $M = \bigoplus_{n\in\mathbb{Z}} M_n$ be a graded $A(R)$-module. We say that M has a Hilbert series if, as an R-module, each graded summand M_n is projective and has well-defined finite rank, i.e., for all $P \in \mathrm{Spec}\, R$, $\mathrm{rank}_{R_P}((M_n)_P)$ is constant across $\mathrm{Spec}\, R$. For such a module M, we write $\mathrm{rank}_R(M_n)$ for $\mathrm{rank}_{R_P}((M_n)_P)$ and we define the Hilbert series of such an M by $H_M(t) = \sum_{n\in\mathbb{Z}} \mathrm{rank}_R(M_n) t^n$.

Definition 1.1. [3] Let d be a non-negative integer and R a commutative k-algebra.

(a) Let $r \in \mathbb{N}$. We say that a graded $A(R)$-module $M = \oplus_i M_i$ is a truncated d-linear module of length r if M is cyclic, generated by M_0, and has Hilbert series

$$H_M(t) = \sum_{i=0}^{r-1} \binom{d+i}{d} t^i.$$

(b) We say that a graded $A(R)$-module $M = \oplus_i M_i$ is a d-linear module if M is cyclic, generated by M_0, and has Hilbert series

$$H_M(t) = \frac{1}{(1-t)^{d+1}}.$$

Clearly any d-linear module may be "truncated" to a truncated d-linear module of any required length. However, it is not usually true that a truncated d-linear module may be obtained from a d-linear module by such a truncation. One of the essential questions, when studying linear modules, is which truncated linear modules of length r may be extended to linear modules, or even to truncated linear modules of length $r + 1$.

We refer to 0-linear modules as point modules, 1-linear modules as line modules and 2-linear modules as plane modules, with similar terminology for their truncated analogues.

Definition 1.2. For $d \in \{0, 1, \dots, v\}$ and $r \in \mathbb{N}$, let $F_{d,r}$ denote the functor from k-algebras to sets, which assigns to each k-algebra R the set of all isomorphism classes of truncated d-linear $A(R)$-modules of length r and to each k-algebra homomorphism $f : R \to R'$ the induction functor $\_ \otimes_R R'$. Similarly we write F_d for the functor of d-linear modules. As functors on schemes over k, we will refer to these functors as the functors of flat families of d-linear, or truncated d-linear, modules.

Remark 1.3. In the previous definition, *isomorphism class* refers to degree-zero, graded isomorphisms. By definition, a d-linear $A(R)$-module, M, is cyclic and generated by M_0, so that M is a graded image of $T(R)$. In particular, M_0 is a rank-one locally-free image of $T(R)_0 = R$, and hence $M_0 = R$. This observation "rigidifies" the isomorphism classes. In other words, the automorphism group of such an M is trivial, with every automorphism being given by multiplication by a unit of R.

The main purpose of this section is to generalise a construction of [3] and give a scheme, $\Omega_r(A, d)$, over k which represents the functor of truncated d-linear modules. For the moment, let us fix r and d and recall $v = \dim_k(V)$. We will describe $\Omega_r(A, d)$ as an explicit closed subscheme of a product of Grassmannians by intersecting two other closed subschemes. The reader should keep in mind that the scheme $\Omega_r(A, d)$ will encode the annihilators of generators of the appropriate linear modules. To this end, we introduce the following notation.

Let W denote any finite-dimensional k-vector space and let U denote any subspace of W. We write $\mathbb{G}_n(W)$ for the Grassmannian scheme of n-dimensional subspaces of W, and $\mathbb{G}^n(W)$ for the scheme of subspaces of W of codimension n. There is a natural embedding of the scheme $\mathbb{G}^n(W/U)$ into $\mathbb{G}^n(W)$ as the subscheme of codimension-n subspaces that contain U. We denote this closed subscheme by $\mathbb{G}^n_U(W)$.

Let $C(a, b)$ denote the binomial coefficient $\binom{a}{b}$. For each $r \in \mathbb{N}$ and for each $d \in \{0, 1, \dots, v\}$, let

$$\Upsilon_r(V, d) = \prod_{i=1}^{r} \mathbb{G}^{C(d+i,d)}((V^*)^{\otimes i}).$$

Inside $\Upsilon_r(V,d)$ there is a closed subscheme related to the ideal I defining the algebra A,

$$\Upsilon_r(V,I,d) = \prod_{i=1}^{r} \mathbb{G}_{I_i}^{C(d+i,d)}((V^*)^{\otimes i}),$$

where I_i denotes the homogeneous elements of degree i in I.

Given a *decreasing* sequence of integers $e_1 \geq e_2 \geq \cdots \geq e_r \geq 0$, we denote by

$$\mathbb{F}(e_1, e_2, \ldots, e_r, W)$$

the flag variety of flags $U_1 \subseteq U_2 \subseteq \cdots \subseteq U_r \subseteq W$, where the codimension of the vector subspace U_i in W is e_i. By definition, $\mathbb{F}(e_1, e_2, \ldots, e_r, W)$ is a closed subscheme of the product $\prod_i \mathbb{G}^{e_i}(W)$.

Next define a decreasing sequence of integers k_i by $k_i = C(d+i,d)v^{r-i}$, where $1 \leq i \leq r$. There is a natural scheme morphism $\omega_r : \Upsilon_r(V,d) \to \prod_i \mathbb{G}^{k_i}((V^*)^{\otimes r})$, which, on the closed point $Q = (Q_1, Q_2, \ldots, Q_r)$, is given by

$$\omega_r(Q) = (Q_1 \otimes (V^*)^{\otimes(r-1)},\ Q_2 \otimes (V^*)^{\otimes(r-2)}, \ldots, Q_i \otimes (V^*)^{\otimes(r-i)}, \ldots, Q_r).$$

In fact, ω_r is a product of trivial Segre embeddings, and so is proper.

Let $\Phi_r(V,d) = \omega_r^{-1}(\mathbb{F}(k_1, \ldots, k_r, (V^*)^{\otimes r}))$, which is a closed subscheme of $\Upsilon_r(V,d)$. We define $\Omega_r(A,d)$ as the scheme-theoretic intersection

$$\Omega_r(A,d) = \Phi_r(V,d) \cap \Upsilon_r(V,I,d).$$

In general, $\Omega_r(A,d)$ will fail to be irreducible or reduced. In order to compute the set of closed points of $\Omega_r(A,d)$, consider a closed point $(Q_1, \ldots, Q_r)$ of $\Upsilon_r(A,d)$, i.e., each Q_i is a subspace of $(V^*)^{\otimes i}$ of codimension $C(d+i,d)$. It follows from the definition that the set of closed points of $\Omega_r(A,d)$ is

$$\{(Q_1, \ldots, Q_r) : Q_i \otimes V^* + I_{i+1} \subseteq Q_{i+1},\ 1 \leq i \leq r-1\}.$$

For each $r \geq 2$, let $\pi_{r-1} : \Upsilon_r(V,d) \to \Upsilon_{r-1}(V,d)$ be the projection onto the first $r-1$ factors. Since $\pi_{r-1}(\Omega_r(A,d)) \subseteq \Omega_{r-1}(A,d)$ for all r, we may define

$$\Omega_\infty(A,d) = \varprojlim \Omega_r(A,d),$$

which will play a role analogous to that played by the scheme, Γ, appearing in [3, Corollary 3.13] (see Remark 1.6).

The following result is a straightforward generalisation, from 0-linear to d-linear, of [3, Proposition 3.9] concerning a scheme, Γ_r, which represents the truncated point modules of length r. However, the reader should note that our scheme, $\Omega_r(A,d)$, is *not* the d-linear analogue of Γ_r, although it plays a role analogous to that of Γ_r. Recall that the functor $F_{d,r}$ is defined in Definition 1.2.

Theorem 1.4. *The scheme $\Omega_r(A,d)$ represents the functor $F_{d,r+1}$ of isomorphism classes of truncated d-linear modules of length $r+1$.*

Proof. We fix d and r and let $F = F_{d,r+1}$. Let G be the functor from the category of commutative k-algebras to the category of sets that is represented by $\Omega_r(A,d)$. The k-valued points of G were described above. It suffices to give a natural isomorphism from the functor F to the functor G.

Let R denote a commutative k-algebra and let M denote a d-linear $A(R)$-module of length $r+1$; that is, M represents an element of $F(R)$. Consider M as a $T(R)$-module that is annihilated by $I(R) = I \otimes R$. By Remark 1.3, $M_0 = R$. There exists $u \in M_0$ such that $M = uA(R)$. Let Q denote the annihilator of u in $T(R)$. Since u is unique up to multiplication by a unit of R and since R is central in $A(R)$, it follows that Q is independent of the choice of u. Moreover, Q has the form $Q = \oplus_{i=1}^{\infty} Q_i$, where Q_i is the kernel of the epimorphism $T(R)_i \to M_i$ given by $t \mapsto ut$. Since each M_i, where $0 \le i \le r$, is a projective direct summand of the free R-module $T(R)_i$ of rank $C(d+i,d)$, Q_i is projective with well-defined rank. For $i > r$, we have $Q_i = T(R)_i$ and, for $1 \le i \le r$, Q_i has corank $C(d+i,d)$ in $T(R)_i$. Since M is an $A(R)$-module, we have $Q_i \supseteq I_i(R)$. It follows that the point $(Q_1,\dots,Q_r)$ is an element of $G(R)$. We have defined a transformation, $M \mapsto (Q_1,\dots,Q_r)$, from F to G; naturality is clear.

Conversely, let $(Q_1,\dots,Q_r)$ be an R-valued point of $\Omega_r(A,d)$; that is, $(Q_1,\dots,Q_r) \in G(R)$. Let $Q = \oplus_{i=1}^{\infty} Q_i$ where, for $i > r$, we take $Q_i = T(R)_i$. The structure of $\Omega_r(A,d)$ implies that Q is a right ideal of $T(R)$ containing $I(R)$. Let M denote the right $T(R)$-module (and $A(R)$-module), $T(R)/Q$. By definition, M is a cyclic $A(R)$-module generated by M_0 and, for $i > 0$, we have $M_i = T(R)_i/Q_i$. By the definition of G, for $1 \le i \le r$, Q_i is a direct summand of $T(R)_i$ of well-defined corank $C(d+i,d)$. Thus, every M_i is projective of the appropriate well-defined rank, so M represents a point in $F(R)$. This transformation, $(Q_1,\dots,Q_r) \mapsto M$, from G to F is natural and inverts the previous transformation. ■

Corollary 1.5. *The set of closed points of $\Omega_\infty(A,d)$ is in one-to-one correspondence with the set of isomorphism classes of d-linear A-modules, and the scheme $\Omega_\infty(A,d)$ represents the functor of flat families of d-linear A-modules.* ■

We wish to compare the constructions given above to those of [3, §3], where the functors $F_{0,r+1}$ and F_0 are represented by different schemes than those used herein.

For any subspace U of a vector space W, let $U^\perp$ denote the orthogonal space to U in W^*. For all of our finite-dimensional vector spaces W, we identify W^{**} with W. Let

$$\Upsilon_r^\perp(V,d) = \prod_{i=1}^{r} \mathbb{G}_{C(d+i,d)}(V^{\otimes i}).$$

There are natural isomorphisms of schemes $\mathbb{G}^n(W) \to \mathbb{G}_n(W^*)$ and $\mathbb{G}_n(W) \to \mathbb{G}^n(W^*)$, both given on closed points by $Q \mapsto Q^\perp$. We will denote these isomorphisms by $\perp$ and extend them to isomorphisms between $\Upsilon_r^\perp(V,d)$ and $\Upsilon_r(V,d)$. Now we take $\Omega_r^\perp(A,d)$ to be the image (or preimage) of $\Omega_r(A,d)$ in $\Upsilon_r^\perp(V,d)$. The set of closed points of $\Upsilon_r^\perp(V,d)$ may be obtained by applying $\perp$ to the closed points in $\Omega_r(A,d)$, which yields

$$\{(q_1,\ldots,q_r) : q_i \otimes V \cap I_{i+1}^\perp \;\supseteq\; q_{i+1},\ 1 \le i \le r-1\}.$$

Finally, following [3], let Γ_r denote the locus of zeros in $\mathbb{P}(V)^r$ of $I_r \subset (V^*)^{\otimes r}$.

Remark 1.6. We focus on the case $d = 0$. It is not difficult to see that there is an isomorphism between $\Omega_r^\perp(A,0)$ and Γ_r, which may be described on closed points as follows. If $(q_1,\ldots,q_r)$ is a closed point of $\Omega_r^\perp(A,0)$, then $\dim_k(q_i) = 1$ for all i. In this case, $q_{i+1} = q_i \otimes p_{i+1}$ for some closed point p_{i+1} of $G_1(V) = \mathbb{P}(V)$ where $1 \le i \le r-1$. Hence the map is given by $(q_1,q_2,\ldots,q_r) \mapsto (q_1,p_2,\ldots,p_r)$. From this formula, one may see that the inverse isomorphism $\Gamma_r \to \Omega_r^\perp(A,0)$ is the restriction of a product of Segre embeddings. It is proved in [3, Proposition 3.9] that Γ_r represents the functor of flat families of truncated point modules of length $r+1$, whereas, herein, the isomorphic scheme $\Omega_r(A,0)$ represents that functor. It is explained in [3] that the points of the scheme Γ_r encode the action of $A(R)$ on the appropriate module. Our scheme, $\Omega_r(A,0)$, encodes the annihilator of the cyclic generator of the appropriate module. Both constructions are dependent on the rigidification of the modules given by the condition $M_0 = R$. It is not clear, however, how one may generalise the scheme Γ_r to the higher-dimensional d-linear cases.

The scheme, $\Omega_\infty(A,0)$, which represents the functor of point modules of A is called, in [31], the point scheme of A. For many quadratic algebras, it suffices to compute $\Omega_2(A,0)$ in order to compute the point scheme ([3, 10, 13, 16, 17, 21, 29, 33]).

Definition 1.7. We call $\Omega_\infty(A,1)$ the *line scheme*, or the *right line scheme*, of A.

In the next section, additional hypotheses will be assumed on A in order to allow explicit computation of the line scheme.

The following technical result generalises [3, Proposition 3.6]. The argument of the proof will be used several times in the sequel.

Lemma 1.8. *The map $\bar{\pi}_{r-1} : \Omega_r(A,d) \to \Upsilon_{r-1}(V,d)$, which is the restriction of the projection map $\pi_{r-1} : \Upsilon_r(V,d) \to \Upsilon_{r-1}(V,d)$, is a proper morphism whose fibres are Grassmannian schemes. If $\bar{\pi}_{r-1}$ is injective on closed points, then it is a closed immersion.*

Proof. Throughout the proof, r is fixed and π denotes $\bar{\pi}_{r-1}$. It is clear that π is proper.

Choose a closed point $P = (Q_1, \ldots, Q_{r-1})$ in $\Upsilon_{r-1}(V,d)$. If $P \notin \Omega_{r-1}(A,d)$, then $\pi^{-1}(P)$ is empty, so we may assume $P \in \Omega_{r-1}(A,d)$. In this case,

$$\pi^{-1}(P) = \{(Q_1, \ldots, Q_r) : Q_r \in \mathbb{G}^{C(d+r,d)}((V^*)^{\otimes r}),\ Q_{r-1} \otimes V^* + I_r \subseteq Q_r\},$$

which is isomorphic to the Grassmannian scheme $\mathbb{G}^{C(d+r,d)}(\,(V^*)^{\otimes r}/(Q_{r-1} \otimes V^* + I_r)\,)$.

Suppose that π is injective on closed points. In particular, π is a proper, quasi-finite morphism. By a standard application of Zariski's Main Theorem in [14], such a map is a finite morphism; that is, π is a finite morphism whose nonempty fibres over closed points are 0-dimensional Grassmannian schemes, $\operatorname{Spec} k$. The result follows. ■

2. Coordinate-free Descriptions of Schemes of Line Modules

By Corollary 1.5, the collection of flat families of line modules of the algebra A carries a natural scheme structure, namely $\Omega_\infty(A,1)$. In this section we impose several conditions on A and on its linear modules which allow us to reduce the computation of the line scheme, $\Omega_\infty(A,1)$, to the computation of various schemes isomorphic to $\Omega_2(A,1)$. We will obtain several different descriptions of $\Omega_2(A,1)$, each with its own use.

Recall that A is a graded algebra of the form $T(V^*)/I$. We identify A_1 with $T(V^*)_1$ and with V^*. In the sequel, additional hypotheses on A will be useful.

Condition 2.1. The Hilbert series $H_A(t) = (1-t)^{-4}$.

Condition 2.2. The algebra A is a domain.

Condition 2.3. If M is any right plane module over A, then M is graded-homogeneous and critical with respect to Gelfand-Kirillov dimension; that is, any nonzero graded submodule (respectively, proper quotient) has GK-dimension three (respectively, two).

If A satisfies Conditions 2.1 and 2.2, then a description of the set of isomorphism classes of right plane modules of A is easily obtained. Indeed, let M be any right plane module. Since $\dim_k(A_1) = 4$ and $\dim_k(M_1) = 3$, it follows that $\mathrm{Ann}_{A_1}(M_0) = ku$ for some nonzero $u \in A_1$. However, A/uA and M have the same Hilbert series, $(1-t)^{-3}$, so the canonical epimorphism $A/uA \twoheadrightarrow M$ is an isomorphism. Hence, every right plane module is isomorphic to A/uA for some nonzero $u \in A_1 = V^*$, and so we may identify the scheme of right plane modules $\Omega_\infty(A,2)$ with $\mathbb{P}(V^*) = \mathbb{G}^3(V^*)$.

In order to view the line scheme as a subscheme of $\mathbb{G}^2(V^*)$, consider $\Omega_2(A,1) \subseteq \Upsilon_2(V,1) = \mathbb{G}^2(V^*) \times \mathbb{G}^3(V^* \otimes V^*)$. Let $\mathcal{L}^\perp$ denote the image of the map $\bar{\pi}_1 : \Omega_2(A,1) \to \Upsilon_1(V,1) = \mathbb{G}^2(V^*)$ defined in Lemma 1.8. A point Q in $\mathbb{G}^2(V^*)$ belongs to the underlying set of $\mathcal{L}^\perp$ if and only if $\dim_k(Q \otimes V^* + I_2) \le 16 - 3 = 13$. However, $\dim_k(Q \otimes V^*) = 8$ and $\dim_k(I_2) = 6$, so $\mathcal{L}^\perp$ may be described as $\mathcal{L}^\perp = \{Q \in \mathbb{G}^2(V^*) : \dim_k(Q \otimes V^* \cap I_2) \ge 1\}$.

If M is a graded A-module, then we define a shifted module, $M[n]$, by $M[n]_i = M_{n+i}$, where $n \in \mathbb{Z}$. The following technical result, in a different form, may be found in [13, Page 55]. We include a proof for completeness.

Lemma 2.4. *Suppose A satisfies Conditions 2.1-2.3. Let $\mathcal{L}^\perp$ denote the image of the morphism $\bar{\pi}_1 : \Omega_2(A,1) \to \mathbb{G}^2(V^*)$ given in Lemma 1.8.*

(i) *If $r \ge 2$, then $\bar{\pi}_r : \Omega_{r+1}(A,1) \to \Omega_r(A,1)$ is an isomorphism.*

(ii) *The morphism $\bar{\pi}_1 : \Omega_2(A,1) \to \mathcal{L}^\perp$ is an isomorphism.*

(iii) *If $Q_1 \in \mathcal{L}^\perp$ is a closed point, then A/Q_1A is a line module, and any line module L is isomorphic to A/Q_1A for some closed point $Q_1 \in \mathcal{L}^\perp$.*

(iv) *If $Q_1 \in \mathcal{L}^\perp$ is a closed point, then $\dim_k(Q_1 \otimes V^* \cap I_2) = 1$.*

Proof. The equations which define the scheme $\Omega_r(A,1)$ are a subset of those which define the scheme $\Omega_{r+1}(A,1)$. Therefore, by Lemma 1.8, part (i) would follow if we establish that $\bar{\pi}_r : \Omega_{r+1}(A,1) \to \Omega_r(A,1)$ is bijective on closed points. Moreover, (ii) would be proved if we establish that $\bar{\pi}_1 : \Omega_2(A,1) \to \mathcal{L}^\perp$ is bijective on closed points.

Fix $Q_1 \in \mathcal{L}^\perp \subset \mathbb{G}^2(V^*)$ and choose a basis $\{u, v\} \subset V^*$ for Q_1. Set $L(Q_1) = A/Q_1A = A/(uA + vA)$. We may choose $a, b \in V^*$ such that $0 \neq u \otimes a + v \otimes b \in I_2$, since we have $\dim_k(Q_1 \otimes V^* \cap I_2) \geq 1$. By Condition 2.2, A is a domain, so neither a nor b is zero. Let M be the plane module A/vA and let $\bar{u}$ denote the image of u in M. By Condition 2.3, we have $\mathrm{GKdim}(\bar{u}A) = 3$. However, $\bar{u}a = 0$ in M, so $\bar{u}A$ is a quotient of the shifted plane module $(A/aA)[-1]$. By Condition 2.3, it follows that $\bar{u}A \cong (A/aA)[-1]$. In particular, since $L(Q_1) \cong M/\bar{u}A$, its Hilbert series is $(1-t)^{-2}$, so $L(Q_1)$ is a line module. Hence, for each $r \in \mathbb{N}$, the module $L(Q_1)(r) := L(Q_1)/(\bigoplus_{n=r+1}^{\infty} L(Q_1)_n)$ is a truncated line module of length $r+1$.

On the other hand, let $L(r)$ denote any truncated line module of length $r+1$, and let $Q_1 = \mathrm{Ann}_{A_1}(v_L) \subset V^*$, where $L(r) = v_LA$ and $\deg(v_L) = 0$. Since $\dim_k(Q_1) = \dim_k(A_1) - \dim_k(L(r)_1) = 2$, and since $\dim_k(Q_1 \otimes V^* + I_2) \leq \dim_k(T(V^*)_2) - \dim_k(L(r)_2) = 13$, it follows that $\dim_k(Q_1 \otimes V^* \cap I_2) \geq 1$. Thus, $Q_1 \in \mathcal{L}^\perp$ and $L(Q_1)$ is a line module. However, $L(r)$ is a quotient of $L(Q_1)$ and, by comparing Hilbert series, we have $L(r) \cong L(Q_1)(r)$.

We have shown that the compositions $\bar{\pi}_1 \circ \bar{\pi}_2 \circ \cdots \circ \bar{\pi}_r : \Omega_{r+1}(A,1) \to \mathcal{L}^\perp$ are all bijections on the level of closed points with pointwise inverses which are $Q_1 \mapsto L(Q_1)(r+1)$. The annihilator in $T(V^*)_2$ of $L(Q_1)$ is precisely $Q_1 \otimes V^* + I_2$ which has dimension thirteen. This proves assertion (iii), and shows that $\bar{\pi}_1 : \Omega_2(A,1) \to \mathcal{L}^\perp$ is bijective with inverse $Q_1 \mapsto (Q_1, Q_1 \otimes V^* + I_2)$, which completes the proof. ■

Lemma 2.4 reduces the problem of computing the line scheme, $\Omega_\infty(A,1)$, of A to that of computing either $\Omega_2(A,1)$ or $\mathcal{L}^\perp$. However, if one wishes to compare the line scheme of A to the point scheme of A, then the natural scheme to compute is the image of $\mathcal{L}^\perp$ in $\mathbb{G}_2(V)$ under the orthogonal isomorphism $\perp: \mathbb{G}^2(V^*) \to \mathbb{G}_2(V)$, namely the isomorphic scheme

$$\mathcal{L} = \{q \in \mathbb{G}_2(V) : \dim_k(q \otimes V \cap I_2^\perp) \geq 3\}.$$

We will return to this description of the line scheme in §3.

Another description of the line scheme is within the projective space $\mathbb{P}(I_2)$ and should generally yield the simplest way to explicitly compute the line scheme.

If $a \otimes b \in V^* \otimes V^*$ and if $v \in V$, then we write $v(a \otimes b) = a(v)b \in V^*$ and $(a \otimes b)v = ab(v) \in V^*$. Extending linearly, we obtain two isomorphisms of $\mathbb{P}(V^* \otimes V^*)$ to the projective space $\mathbb{P}(\mathrm{Hom}_k(V, V^*))$. Using either of these isomorphisms, we may unambiguously refer to the rank of an element of $\mathbb{P}(V^* \otimes V^*)$.

Let $\mathcal{H}$ denote the scheme-theoretic intersection in $\mathbb{P}(V^* \otimes V^*)$ of the subscheme $\mathbb{P}(I_2)$ with the scheme of all tensors of rank two, and let $\bar{\mathcal{H}}$ denote the closure of $\mathcal{H}$ in $\mathbb{P}(V^* \otimes V^*)$; that is, $\bar{\mathcal{H}}$ is the intersection of $\mathbb{P}(I_2)$ with the closed subscheme of tensors of rank at most two.

Lemma 2.5. *Let $\phi : \mathcal{H} \to \mathbb{G}^2(V^*)$ be the morphism given by $\phi(\lambda) = \mathrm{Ker}(\lambda)^\perp$. If Conditions* 2.1-2.3 *hold, then $\mathcal{H} = \bar{\mathcal{H}}$ and ϕ is an isomorphism from $\mathcal{H}$ to the scheme $\mathcal{L}^\perp$.*

Proof. Consider the incidence relation

$$\Lambda = \{(Q, \lambda) \in \mathbb{G}^2(V^*) \times \mathbb{P}(V^* \otimes V^*) : \lambda \subset Q \otimes V^* \cap I_2\}.$$

Let $\Pi_1 : \Lambda \to \mathbb{G}^2(V^*)$ and $\Pi_2 : \Lambda \to \mathbb{P}(V^* \otimes V^*)$ be the restrictions of the canonical projections. The fibres of Π_1 are projective spaces of the form $\mathbb{P}(Q \otimes V^* \cap I_2)$. If $\lambda \in \mathbb{P}(V^* \otimes V^*)$, then there are three possibilities for the fibre $\Pi_2^{-1}(\lambda)$. If $\mathrm{rank}(\lambda) \geq 3$ or if $\lambda \notin \mathbb{P}(I_2)$, then the fibre is empty. If $\mathrm{rank}(\lambda) = 1$ and $\lambda \in \mathbb{P}(I_2)$, then $\lambda = a \otimes b$ for some nonzero $a, b \in \mathbb{P}(V^*)$. In this case, the fibre over λ is $\{(Q, a \otimes b) : a \subset Q\} \cong \mathbb{P}^2$. However, if $\mathrm{rank}(\lambda) = 2$ and $\lambda \in \mathbb{P}(I_2)$, then $\Pi_2^{-1}(\lambda) = \{(Q, \lambda) : \lambda \subset Q \otimes V^*\} = \{(Q, \lambda) : Q^\perp = \mathrm{Ker}(\lambda)\} \cong \mathbb{P}^0$.

By Condition 2.2, no element of I_2 has rank one. Thus, $\mathcal{H} = \bar{\mathcal{H}}$ and Π_2 is injective on closed points. Lemma 2.4 implies that Π_1 is injective on closed points. By the argument of Lemma 1.8, we have that Π_1 and Π_2 are both closed immersions. Moreover, the image of Π_1 is clearly $\mathcal{L}^\perp$ and the image of Π_2 is $\mathcal{H}$, so the incidence relation Λ is the graph of $\psi : \mathbb{G}^2(V^*) \to \mathcal{H}$ where $\psi(Q) = Q \otimes V^* \cap I_2$. The result follows since ψ is the inverse of ϕ. ■

If Conditions 2.1-2.3 hold, then the subscheme of $\mathbb{P}(V^* \otimes V^*)$ of tensors of rank at most two has dimension eleven and $\mathbb{P}(I_2)$ has dimension five, so we obtain the following result.

Corollary 2.6. *If Conditions* 2.1-2.3 *hold, then the irreducible components of the line scheme of A have dimension at least one.* ■

Similar definitions and results may be given for the *left* linear modules of A. In the following result, we identify $\mathbb{P}(V^* \otimes V^*)$ with $\mathbb{P}(\text{Hom}(V, V^*))$ via $(a \otimes b)(v) = ab(v)$ where $a, b \in V^*$, $v \in V$, and exploit the left-right symmetry of $\mathcal{H}$.

Corollary 2.7. *In addition to Conditions 2.1-2.3, assume that the left plane modules of A satisfy the left analogue of Condition 2.3. If $\mathcal{L}^{\perp}_{left} \subset \mathbb{G}^2(V^*)$ denotes the left analogue of the scheme $\mathcal{L}^{\perp}$, then $\mathcal{L}^{\perp}_{left}$ represents the functor of left line modules of A, and the schemes $\mathcal{L}^{\perp}$ and $\mathcal{L}^{\perp}_{left}$ are isomorphic via the isomorphism $Q \mapsto Ker(\lambda)^{\perp}$, where $\lambda = Q \otimes V^* \cap I_2$.* ■

In the sequel, it will be of interest to consider $\gamma_2 : \Omega_2(A, 1) \to \mathbb{G}^3(V^* \otimes V^*)$ which is the restriction of the second-term projection $\Upsilon_2(V, 1) \to \mathbb{G}^3(V^* \otimes V^*)$ to $\Omega_2(A, 1)$. Let $\Delta_2^{\perp}$ denote the image of γ_2. The fibres of the morphism γ_2 over a closed point $Q_2 \in \mathbb{G}^3(V^* \otimes V^*)$ may be described as follows. If $I_2 \not\subset Q_2$, then $\gamma_2^{-1}(Q_2)$ is empty. However, if $I_2 \subset Q_2$, then $\gamma_2^{-1}(Q_2) = \{(Q_1, Q_2) : Q_1 \otimes V^* \subset Q_2\} \cong \mathbb{G}^d(W(Q_2))$, where $W(Q_2) = \{\alpha \in V^* : k\alpha \otimes V^* \subset Q_2\}$ and $d = 2 - \text{codim}(W(Q_2))$.

Condition 2.8. If L is a right line module of A, then L has no finite-dimensional graded submodules.

Lemma 2.9. *Assume A satisfies Conditions 2.1-2.3. If γ_2 is injective on closed points, then $\gamma_2 : \Omega_2(A, 1) \to \mathbb{G}^3(V^* \otimes V^*)$ is a closed immersion with image $\Delta_2^{\perp}$. If A also satisfies Condition 2.8, then γ_2 is an isomorphism.*

Proof. The first statement is proved using the argument in Lemma 1.8.

Suppose that (Q_1, Q_2) and (Q_1', Q_2) are two distinct closed points in $\Omega_2(A, 1)$. In particular, $Q_1, Q_1' \in \mathcal{L}^{\perp}$ and $Q_1, Q_1' \subset W(Q_2)$. By Lemma 2.4, we have

$$Q_1 \otimes V^* + I_2 = Q_2 = Q_1' \otimes V^* + I_2. \qquad (*)$$

If $Q_1 \cap Q_1' = 0$, then $W(Q_2) = V^*$, which contradicts the fact that $\dim_k(Q_2) = 13$. Thus, $\dim_k(Q_1 \cap Q_1') = 1$, so we may choose a basis $\{a, b\}$ for Q_1 and a basis $\{a, b'\}$ for Q_1'. Let M denote the plane module A/aA, let L denote the line module A/Q_1A, let $\bar{b}$ and $\bar{b}'$ denote the images of b and b' in M, and write $f : M \to L$ for the canonical homomorphism. It follows that $\text{Ker}(f) = \bar{b}A$ and $f(\bar{b}') \in L_1$ is nonzero. However, $(*)$ implies that $f(\bar{b}')A_1 = 0$ in L, which contradicts Condition 2.8. ■

In the sequel, we will also use the scheme

$$\Delta_2 = \{q_2 \in \mathbb{G}_3(V \otimes V) : q_2 \subseteq q_1 \otimes V \cap I_2^{\perp} \text{ for some } q_1 \in \mathbb{G}_2(V)\},$$

which is isomorphic to $\Delta_2^{\perp}$ under the orthogonal isomorphism $\perp: \mathbb{G}^3(V^* \otimes V^*) \to \mathbb{G}_3(V \otimes V)$.

Remark 2.10. In subsequent sections, the algebras considered will be quadratic, noetherian, Auslander-regular algebras of global dimension four which satisfy the Cohen-Macaulay property. We will refer to such an algebra as a quantum $\mathbb{P}^3$. Suppose A is such an algebra. It is standard that A satisfies Condition 2.1 (c.f., [17, Lemma 1.3]), and, by [4, 12], A satisfies Condition 2.2. Moreover, by [13], plane modules and line modules of A are homogeneous and critical with respect to GK-dimension, so A satisfies Conditions 2.3 and 2.8. Thus, the line scheme of a quantum $\mathbb{P}^3$ satisfies all the results of this section. By [17, Theorem 1.4], the scheme Γ_2, discussed in Remark 1.6, is the graph of an automorphism of a scheme $\mathcal{P} \subseteq \mathbb{P}(V)$. It follows, as in [3], that the schemes Γ_r, where $r \geq 2$, are all isomorphic, so, by Remark 1.6, the schemes $\Omega_r(A, 0)$, where $r \geq 2$, are all isomorphic. In particular,

$$\Omega_2(A, 0) = \{(Q_1, Q_2) \in \mathbb{G}^1(V^*) \times \mathbb{G}^1(V^* \otimes V^*) : Q_1 \otimes V^* + I_2 = Q_2\},$$

where A is a quantum $\mathbb{P}^3$. In this case, we may describe the closed points of $\mathcal{P}$ as $\{p \in \mathbb{P}(V) : \dim_k(p^{\perp} \otimes V^* + I_2) = 15\}$. This description of $\mathcal{P}$ should be compared with the description of closed points given for $\mathcal{L}^{\perp}$ on page 10.

In summary, under Conditions 2.1-2.3 and Condition 2.8, the (right) line scheme of A is given by any of the following isomorphic schemes:

- $\mathcal{L} = \{q \in \mathbb{G}_2(V) : \dim_k(q \otimes V \cap I_2^{\perp}) \geq 3\}$,
- $\Delta_2 = \{q_2 \in \mathbb{G}_3(V \otimes V) : q_2 = q_1 \otimes V \cap I_2^{\perp} \text{ for some } q_1 \in \mathbb{G}_2(V)\}$,
- $\mathcal{L}^{\perp} = \{Q \in \mathbb{G}^2(V^*) : \dim_k(Q \otimes V^* + I_2) \leq 13\}$,
- $\Delta_2^{\perp} = \{Q_2 \in \mathbb{G}^3(V^* \otimes V^*) : Q_2 = Q_1 \otimes V^* + I_2 \text{ for some } Q_1 \in \mathbb{G}^2(V^*)\}$,
- $\mathcal{H} = \{\lambda \in \mathbb{P}(V^* \otimes V^*) : \text{rank}(\lambda) \leq 2 \text{ and } \lambda \subset I_2\}$.

Each of these schemes is a closed subscheme of the appropriate Grassmannian space, and the inequalities which define $\mathcal{L}$, $\mathcal{L}^{\perp}$ and $\mathcal{H}$ may be taken to be equalities. The schemes $\mathcal{L}$ and $\mathcal{H}$ will be the most useful in the sequel.

3. Incidence Relations between Linear Modules

Throughout this section we assume that A is a quantum $\mathbb{P}^3$ as defined in Remark 2.10. In this case, line modules are parametrized by the scheme $\mathcal{L} \subset \mathbb{G}_2(V)$, point modules by the scheme $\mathcal{P} \subset \mathbb{P}(V)$, which is defined in Remark 2.10, and Γ_2 is the graph of an automorphism of $\mathcal{P}$. Moreover, by [10, 13], in this setting, point modules and line modules are determined by degree-one elements of A; in particular, if $q \in \mathcal{L}$, then we write $L(q)$ for the right line module $A/q^{\perp}A$, and if $p \in \mathcal{P}$, then we write $M(p)$ for the right point module $A/p^{\perp}A$. We remark that if $p \in \mathcal{P}$ and $q \in \mathcal{L}$, then $M(p)$ is a quotient of $L(q)$ if and only if $p \in \mathbb{P}(q)$.

Given the right point module $M(p)$, we will write $\mathcal{L}_p$ for the subscheme of $\mathcal{L}$ consisting of those line modules $L(q)$ such that $M(p)$ is a quotient of $L(q)$.

Proposition 3.1. *Every point module over A is a quotient of some line module; that is, $\mathcal{L}_p$ is nonempty for every $p \in \mathcal{P}$.*

Proof. Suppose that $p \in \mathcal{P}$ and define $\Lambda_p = \{\zeta \in \mathbb{P}(V^* \otimes V^*) : \mathrm{rank}(\zeta) \leq 2,\ p\zeta = 0\}$. For any $q \in \mathcal{L}$, let $\zeta_q = q^{\perp} \otimes V^* \cap I_2 \in \mathcal{H} \subset \mathbb{P}(I_2)$. It follows that $M(p)$ is a quotient of $L(q)$ if and only if $q \in \mathcal{L}$ and $\zeta_q \in \Lambda_p$. Hence it suffices to prove that $\Lambda_p \cap \mathcal{H}$ is nonempty. The scheme Λ_p is a closed 9-dimensional subscheme of $\mathbb{P}(p^{\perp} \otimes V^* + I_2)$ and the scheme $\mathbb{P}(I_2)$ has dimension five. By Remark 2.10, $\mathbb{P}(p^{\perp} \otimes V^* + I_2) \cong \mathbb{P}^{14}$, from which it follows that $\Lambda_p \cap \mathbb{P}(I_2)$ has nonnegative dimension and so is nonempty. Since $\Lambda_p \cap \mathcal{H} = \Lambda_p \cap \mathbb{P}(I_2)$, the result follows. ■

Remark 3.2. Let $p \in \mathcal{P}$ and suppose that there are at most finitely many right line modules which cover $M(p)$. It follows that the intersection $\Lambda_p \cap \mathbb{P}(I_2)$, used in the proof of Proposition 3.1, has minimal dimension, namely zero. By Bézout's theorem, the number of points in this intersection, counting multiplicity, is the degree of Λ_p as a subscheme of $\mathbb{P}(p^{\perp} \otimes V^* + I_2)$, and, by [6, Example 19.10], this degree is six. We conclude that if the point module $M(p)$ is covered by at most finitely many line modules, then it is covered by six line modules, counting multiplicity. At this time, the only algebra known to exhibit this phenomenon has a point scheme consisting of twenty distinct points and a 1-dimensional line scheme ([15, 19]).

Recall that, by Remark 2.10, line modules over a quantum $\mathbb{P}^3$ are homogeneous. As noted earlier, if M is a graded A-module, then a shifted module, $M[n]$, may be defined by $M[n]_i = M_{n+i}$, where $n \in \mathbb{Z}$.

Lemma 3.3. *If L is a line module over A, then* $\dim_k(\mathrm{Ann}_{A_1}(v)) \leq 2$ *for all* $v \in L_n$, $n \in \mathbb{N}$.

Proof. Let L be a line module. For every $v \in L_n$, for all $n \in \mathbb{N}$, we have $\dim_k(\mathrm{Ann}_{A_1}(v)) \leq 3$, by homogeneity of L. Hence, if the result were false, there would exist $v \in L_n$, for some $n \in \mathbb{N}$, such that $\dim_k(\mathrm{Ann}_{A_1}(v)) = 3$. In this case, $\mathrm{Ann}_{A_1}(v) = p^\perp$ for some $p \in \mathbb{P}(V)$, so that $A/p^\perp A \twoheadrightarrow vA[n]$. Thus, since L is homogeneous, $\dim_k(p^\perp A_1) < \dim_k(A_2) = 10$. Hence, $A/p^\perp A$ would be a point module, so that $\mathrm{GKdim}(vA) \leq 1$. This would contradict the homogeneity of L. ■

Proposition 3.4. *Suppose $q \in \mathcal{L}$. If $p_1, \ldots, p_4 \in \mathcal{P}$ are distinct points with the property that $p_i \in \mathbb{P}(q)$, $1 \leq i \leq 4$, then $\mathbb{P}(q) \subseteq \mathcal{P}$. In this case, there exists $q' \in \mathcal{L}$ such that for every $p \in \mathbb{P}(q)$ there is a short exact sequence*

$$0 \longrightarrow L(q')[-1] \longrightarrow L(q) \longrightarrow M(p) \longrightarrow 0,$$

that is, up to isomorphism, the kernel of the map $L(q) \to M(p)$ does not vary with $p \in \mathbb{P}(q)$.

Proof. By considering Hilbert series, the kernel of a graded epimorphism from a line module to a point module is a shifted line module. Hence, for each $i = 1, \ldots, 4$, we may choose $q_i \in \mathcal{L}$ such that $L(q_i)[-1]$ is isomorphic to the kernel of $M(q) \twoheadrightarrow M(p_i)$. For each $i = 1, \ldots, 4$, consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L(q_i)[-1] & \longrightarrow & L(q) & \longrightarrow & M(p_i) & \longrightarrow & 0 \\ & & & & \downarrow & & & & \\ & & & & M(p_j) & & & & \end{array}$$

where $j \neq i$. Since $M(p_j)_{\geq 1} \cong M(\sigma(p_j))[-1]$, where $\sigma \in \mathrm{Aut}(\mathcal{P})$ is the automorphism determined by Γ_2, we have that $L(q_i) \twoheadrightarrow M(\sigma(p_j))$ for all $j \neq i$, so that $\sigma(p_j) \in \mathbb{P}(q_i)$ for all $j \neq i$. It follows that $q_i = q_j$ for all i, j. Since $\dim_k(L_1) = 2$ and since the p_i are distinct, we have that $\mathrm{Ann}_{A_1}(L(q)_1) = q_i^\perp$ for all $i = 1, \ldots, 4$. By Lemma 3.3, $\dim_k(\mathrm{Ann}_{A_1}(v)) = 2$ for all $v \in L(q)_1$, so that $L(q)/vA$ is isomorphic to a point module for all $v \in L(q)_1$, which completes the proof. ■

4. Geometrically Defined Algebras

Having attached to the algebra $A = T(V^*)/I$ the geometric data $\Omega_r(A, d)$, the question arises as to how much geometric data is required before one is able to recover A. For example, it is a consequence of [3, 4] that the defining relations of an Artin-Schelter regular algebra of global dimension three (generated by homogeneous degree-one elements) may be recovered from the data giving the point scheme. If such an algebra has three generators, then it is quadratic and the relations are exactly the 2-forms which vanish on the scheme Γ_2. For quadratic Artin-Schelter regular algebras of global dimension four, it was shown in [17, 31] that Γ_2 does not, in general, determine the defining relations of the algebra. Examples of this phenomenon will be discussed at the end of this section, but we will demonstrate that these examples are not generic.

The following result substantially simplifies the proofs of the main results in [16].

Theorem 4.1. *Let $T(V^*)/I$ be a quadratic algebra on four generators with six defining relations, and let $\Gamma_2 \subset \mathbb{P}(V) \times \mathbb{P}(V)$ denote the zero locus of I_2. If $\dim(\Gamma_2) = 0$, then*

$$I_2 = \{f \in V^* \otimes V^* : f|_{\Gamma_2} = 0\}.$$

Proof. We identify Γ_2 with its image under the Segre embedding $\mathbb{P}(V) \times \mathbb{P}(V) \hookrightarrow \mathbb{P}(V \otimes V)$. Since the image of $\mathbb{P}(V)\times\mathbb{P}(V)$ is the scheme Λ_1 of rank-one elements, we have $\Gamma_2 = \mathbb{P}(I_2^\perp)\cap\Lambda_1$. However, $\dim(\mathbb{P}(I_2^\perp)) = 9$ and $\dim(\Lambda_1) = 6$, so it follows that Γ_2 has the minimal possible dimension, namely zero. Let S and R denote the homogeneous coordinate rings of $\mathbb{P}(V \otimes V)$ and Λ_1 respectively, and let $\rho : S \to R$ denote the canonical epimorphism. We write J for the ideal of S generated by $I_2^\perp$ and set $J_R = \rho(J)$. If the result were false, then J_R would not be saturated, so J_R would have the irrelevant ideal of R as an associated prime.

Since $\dim(\Gamma_2) = 0$, we have $\text{height}(J_R) = 6$, which is the minimal number of generators of J_R. By [5, Theorem 18.18], R is a Cohen-Macaulay ring, and hence, by Macaulay's Unmixedness Theorem ([5, Corollary 18.14]), J_R has no embedded primes; in particular, the irrelevant ideal of R is not an associated prime of J_R. ■

Remark 4.2. The proof of Theorem 4.1 shows that the ideal J is saturated. Moreover, by work of M. Van den Bergh, the zero locus of the defining relations of a quadratic algebra on four generators with six *generic* relations is finite. Thus, Theorem 4.1 holds for any quadratic algebra on four generators which has six *generic* relations.

In Theorem 4.3 below, almost the same argument as the one used to prove Theorem 4.1 shows that the line scheme of a quantum $\mathbb{P}^3$ determines the defining relations whenever the line scheme has minimal dimension. We thank J. T. Stafford for pointing out to us that the generic member of the class of four-dimensional regular algebras classified in [24] has infinite point scheme and a line scheme of minimal dimension.

Recall that the scheme $\mathcal{H}$ from §2 is the intersection of $\mathbb{P}(I_2)$ with Λ_2, the scheme of elements in $\mathbb{P}(V^* \otimes V^*)$ which have rank at most two, and that $\mathcal{H}$ is isomorphic to the line scheme of A, if A is a quantum $\mathbb{P}^3$.

Theorem 4.3. *Let $A = T(V^*)/I$ be a quantum $\mathbb{P}^3$. If the line scheme of A has dimension one, then $I_2^\perp = \{g \in V \otimes V : g|_{\mathcal{H}} = 0\}$.*

Proof. The scheme Λ_2 is a Cohen-Macaulay scheme of dimension 11 and contains $\mathcal{H}$ as a closed subscheme. Hence, $\mathcal{H} = \mathrm{Proj}\,(R/J)$ where R is the homogeneous coordinate ring of Λ_2 and J is the ideal of R generated by the ten homogeneous degree-one polynomials on $\mathbb{P}(V^* \otimes V^*)$ determined by $I_2^\perp$. Since $\dim(\mathcal{H}) = 1$, the height of J in R is 10, which is the minimal number of generators of J. Hence, we may invoke Macaulay's Unmixedness Theorem as in Theorem 4.1. ■

Of course, for any quantum $\mathbb{P}^3$, we have $I_2^\perp \subseteq \{g \in V \otimes V : g|_{\mathcal{H}} = 0\}$. It is possible to have equality without the hypothesis that the dimension of the line scheme be one and such an algebra is discussed in Example 4.5. In general, it is not sufficient to consider the reduced scheme $\mathcal{H}_{red}$, since $\{g \in V \otimes V : g|_{\mathcal{H}_{\mathrm{red}}} = 0\}$ could be larger than $\{g \in V \otimes V : g|_{\mathcal{H}} = 0\}$.

To compute the relations of the quantum $\mathbb{P}^3$ directly, recall the scheme $\Delta_2 \subset \mathbb{G}_3(V \otimes V)$ from §2. Let $\Psi = \Psi_A$ be the subscheme of $\mathbb{P}(V \otimes V)$ formed by the union of the elements of Δ_2; that is, $\Psi = \Pi_2(\Pi_1^{-1}(\Delta_2))$, where Π_1 and Π_2 are the first and second projections, respectively, from the incidence relation $\Sigma = \{(\delta, U) \in \Delta_2 \times \mathbb{P}(V \otimes V) : U \subset \delta\}$. The following result is immediate from the definitions and Proposition 3.1.

Lemma 4.4. *If A is a quantum $\mathbb{P}^3$, then Γ_2 and Ψ are closed subschemes of $\mathbb{P}(I_2^\perp)$; moreover, every closed point of Γ_2 is a closed point of Ψ.* ■

The result suggests that we should try to recover the defining relations of A by computing both $\{f \in V^* \otimes V^* : f|_{\Gamma_2} = 0\}$ and $\{f \in V^* \otimes V^* : f|_{\Psi} = 0\}$. Example 4.5 demonstrates that it is possible for the latter to produce I_2 even if the former does not. We suspect that Γ_2 is a closed subscheme of Ψ, at least if A is a quantum $\mathbb{P}^3$, in which case it would suffice to compute $\{f \in V^* \otimes V^* : f|_{\Psi} = 0\}$ in order to recover the defining relations of A.

Example 4.5. Fix a 4-dimensional vector space V and let $Q \subset \mathbb{P}(V)$ be a smooth quadric hypersurface. Fix $\tau \in \operatorname{Aut}(Q)$ and $\alpha \in k$ with $\alpha(\alpha^2+1) \neq 0$. Associated to this data is a quantum $\mathbb{P}^3$, namely, $A = A(Q, \tau, \alpha)$ as defined in [17, 31]. It was shown in [17, 31] that, for these algebras, Γ_2 is the graph of the automorphism τ and that the dimension of the space $J = \{f \in V^* \otimes V^* : f|_{\Gamma_2} = 0\}$ is seven. Since $J \supset I_2$, there is an "extra" 2-form vanishing on the point scheme Γ_2. This extra 2-form is represented in A by a quadratic, normal, regular element Ω, and $A/\Omega A$ is the twisted homogeneous coordinate ring of Q determined by τ.

We will show in [18] that the line scheme of this algebra $A(Q, \tau, \alpha)$ is a reduced scheme with three components, namely, the two rulings on Q together with a third 3-dimensional component. The 3-dimensional component of the line scheme corresponds to the 2-dimensional isotropic subspaces of a nondegenerate skew-symmetric bilinear form on V. Since the dimension of the line scheme is not one, Theorem 4.3 does not apply. However, the span, I_2, of the defining relations of A has a basis consisting of rank-two elements of $V^* \otimes V^*$. It follows that $I_2^{\perp} = \{g \in V \otimes V : g|_{\mathcal{H}} = 0\}$ as in Theorem 4.3. On the other hand, it is easily seen that the 2-form Ω does not vanish on the scheme Ψ. Since I_2 vanishes on Ψ, we obtain $I_2 = \{f \in V^* \otimes V^* : f|_{\Psi} = 0 \text{ and } f|_{\Gamma_2} = 0\}$. The reader is referred to [18] for the details of these computations.

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