SCHEMES OF LINE MODULES II

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Abstract. In this sequel to [8], we study the scheme of line modules for several classes of quantum $\mathbb{P}^3$s, including Clifford algebras, homogenized $\mathfrak{sl}(2)$ and algebras associated to smooth quadrics in $\mathbb{P}^3$. We also prove that a quantum $\mathbb{P}^3$ with enough symmetry in its defining relations has a line scheme of dimension at least two, with infinitely many line modules incident to any point module.

1991 *Mathematics Subject Classification.* 16W50, 14A22.
Key words and phrases. Quadratic algebra, point module, line module, point scheme, line scheme.
*The second author was supported in part by NSF grants DMS-9622765 & DMS-9996056.

Introduction and Notation

Let k be a field with $\mathrm{char}(k) \neq 2$ and let V be a four-dimensional k-vector space. Let $T(V)$ be the $\mathbb{N}$-graded tensor algebra on V. For the purposes of this paper, a quantum $\mathbb{P}^3$ is a graded factor algebra $A = T(V^*)/I$ which is Noetherian, quadratic, Auslander-regular of global dimension four and satisfies the Cohen-Macaulay property. The Hilbert series of such an algebra is $H_A(t) = 1/(1-t)^4$ and one may consider it as an algebra on four degree-one generators with six quadratic relations.

For any finite-dimensional vector space W we write $\mathbb{G}_n(W)$ (respectively, $\mathbb{G}^n(W)$) for the Grassmannian scheme of subspaces of W of dimension n (respectively, codimension n). We write $\mathbb{P}(W)$ for $\mathbb{G}_1(W)$. For a subspace $Q \subset W$, we write $Q^\perp$ for the orthogonal complement of Q in W^*. We also use $\perp$ for the canonical isomorphisms $\perp\colon G_n(W) \leftrightarrow G^n(W^*)$.

For the remainder of this introduction, let A denote a quantum $\mathbb{P}^3$. A point module over A is a cyclic graded module M of Hilbert series $H_M(t) = 1/(1-t)$. A line module is a cyclic graded module L of Hilbert series $H_L(t) = 1/(1-t)^2$. From [2], the scheme Γ, of zeroes in $\mathbb{P}(V) \times \mathbb{P}(V) \subset \mathbb{P}(V \otimes V)$ of the defining relations of A, represents the functor of point modules of A, and will be called the *point scheme* of A. In [8] it was shown that there is similarly a subscheme Δ_2 of $\mathbb{G}_3(V \otimes V)$ which represents the functor of line modules of A. Our goal is to continue the study of this line-module scheme, the *line scheme* of A.

From [7], the scheme Γ is the graph of an automorphism σ of a subscheme $\mathcal{P} \subset \mathbb{P}(V)$, where $\mathcal{P}$ has the coordinate-free description: $\mathcal{P} = \{p \in \mathbb{P}(V) : \dim(p^\perp \otimes V^* + I_2) = 15\}$. Likewise, from [8], we have the following coordinate-free descriptions of Δ_2 and several schemes naturally isomorphic to Δ_2.

$$\Delta_2 = \{q_2 \in \mathbb{G}_3(V \otimes V) : q_2 = q_1 \otimes V \cap I_2^\perp \text{ for some } q_1 \in \mathbb{G}_2(V)\},$$
$$\mathcal{L} = \{q \in \mathbb{G}_2(V) : \dim(q \otimes V \cap I_2^\perp) \geq 3\},$$
$$\mathcal{L}^\perp = \{Q \in \mathbb{G}^2(V^*) : \dim(Q \otimes V^* + I_2) \leq 13\},$$
$$\Delta_2^\perp = \{Q_2 \in \mathbb{G}^3(V^* \otimes V^*) : Q_2 = Q_1 \otimes V^* + I_2 \text{ for some } \; Q_1 \in \mathbb{G}^2(V^*)\},$$
$$\mathcal{H} = \{\lambda \in \mathbb{P}(V^* \otimes V^*) : \mathrm{rank}(\lambda) \leq 2 \text{ and } \lambda \subset I_2\}.$$

Each of these schemes is a closed subscheme of the appropriate Grassmannian space. Since A is a quantum $\mathbb{P}^3$, the inequalities which define $\mathcal{L}$, $\mathcal{L}^\perp$ and $\mathcal{H}$ may be taken to be equalities, c.f., [8]. We note that these schemes are all defined for an arbitrary quadratic algebra on four generators with six quadratic relations, but they will not, in that generality, be isomorphic.

For $p \in \mathcal{P}$, we write $M(p)$ for the right point module $A/p^{\perp}A$. Similarly, for $q \in \mathcal{L}$, we write $L(q)$ for the right line module $A/q^{\perp}A$. We say that $M(p)$ lies on $L(q)$ if there is a nonzero graded homomorphism $L(q) \to M(p)$ and we write $\mathcal{L}_p$ for the subscheme of $\mathcal{L}$ of lines q such that $M(p)$ is on $L(q)$.

We begin our computation of line schemes, in §1, by describing the schemes $\mathcal{L}$ and $\mathcal{L}^{\perp}$ in Plücker coordinates. The computations in §1 are for any graded algebra on four generators with six quadratic relations.

In §2 we discuss quantum $\mathbb{P}^3$s which have partially symmetric relations. We prove that for such algebras, the line scheme $\mathcal{L}$ has dimension at least two, and that, for $p \in \mathcal{P}$, the subscheme $\mathcal{L}_p$ has dimension at least one. This is in contrast to the general theorem of [8] that the line scheme has dimension at least one and the subscheme $\mathcal{L}_p$ has dimension at least zero (and multiplicity six when the dimension is zero). We also examine two examples, a Clifford algebra with only one point module and a closely related non-Clifford algebra with only one point module.

In §3 we compute the line schemes of some of the quantum quadrics from [7, 9, 10]. For these quantum $\mathbb{P}^3$s the point scheme contains a quadric surface in $\mathbb{P}^3$. The coordinate ring of quantum 2×2 matrices is such an algebra. In these cases, the line scheme $\mathcal{L}$ is the union of three components in $\mathbb{G}_2(V)$, two of which have dimension one, whereas the other has dimension three. The importance of these examples is the following. For some of these algebras, the point scheme alone does not determine the defining relations. There is, in a sense, an "extra" relation vanishing on the point scheme Γ. We show, however, that the point scheme and the line scheme together determine the relations of the algebra in a natural way.

The homogenization A of the universal enveloping algebra of the Lie algebra $\mathfrak{sl}(2)$ is a quantum $\mathbb{P}^3$ and is discussed in §4. The line scheme $\mathcal{L}$ of A is reducible with two components: a nonreduced plane (counted with multiplicity four) and a reduced "cylinder". The intersection of the two components corresponds to an embedded conic in the point scheme ([4]) which corresponds to the Casimir element.

1. Coordinate Computations

We assume throughout this section that A is a quadratic graded k-algebra of the form $T(V^*)/I$ where $\dim(V) = 4$ and $\dim(I_2) = 6$. We describe the isomorphic schemes

$$\begin{aligned} \mathcal{L} &= \{q \in \mathbb{G}_2(V) : \dim(q \otimes V \cap I_2^\perp) \geq 3\}, \\ \mathcal{L}^\perp &= \{Q \in \mathbb{G}^2(V^*) : \dim(Q \otimes V^* \cap I_2) \geq 1\}, \end{aligned}$$

by using global Plücker coordinates on the appropriate Grassmannian spaces.

Fix a basis $\{e_1, \ldots, e_4\}$ for V and let $\{x_1, \ldots, x_4\}$ denote the dual basis for V^*. For $1 \leq i < j \leq 4$ we have Plücker coordinates M_{ij} on $\mathbb{G}_2(V)$ and N_{ij} on $\mathbb{G}^2(V^*)$ associated to these bases. We write $S_M = k[M_{ij}]$ and $S_N = k[N_{ij}]$ and define the Plücker relations: $p_M = M_{12}M_{34} - M_{13}M_{24} + M_{14}M_{23}$ and $p_N = N_{12}N_{34} - N_{13}N_{24} + N_{14}N_{23}$. With this notation, $\mathbb{G}_2(V) = \text{Proj}\,(S_M/\langle p_M\rangle)$ and $\mathbb{G}^2(V^*) = \text{Proj}\,(S_N/\langle p_N\rangle)$. The orthogonality isomorphism $\perp\colon \mathbb{G}_2(V) \to \mathbb{G}^2(V^*)$ is realized on the coordinates by mapping $N_{12} \mapsto M_{34}$, $N_{13} \mapsto -M_{24}$, $N_{14} \mapsto M_{23}$, $N_{23} \mapsto M_{14}$, $N_{24} \mapsto -M_{13}$ and $N_{34} \mapsto M_{12}$.

Let $\{f_1, \ldots, f_6\}$ be a basis for the space $I_2 \subset V^* \otimes V^*$ and let $\{h_1, \ldots, h_{10}\}$ be a basis for $I_2^\perp \subset V \otimes V$. We write $f = (f_1, \ldots, f_6)^T$ and $h = (h_1, \ldots, h_{10})^T$. We may factor f as $f = M \otimes (x_1, \ldots, x_4)^T$, where M is a 6×4 matrix of 1-forms on V, and similarly $h = N \otimes (e_1, \ldots, e_4)^T$ where N is a 10×4 matrix of 1-forms on V^*. We consider f and h also as linear functions: $f : V \otimes V \to k^6$ and $h : V^* \otimes V^* \to k^{10}$.

To compute $\mathcal{L}^\perp$, fix $Q \in \mathbb{G}^2(V^*)$, and observe that, by definition of h, we have $Q \in \mathcal{L}^\perp$ if and only if $\text{rank}(h|_{Q\otimes V^*}) \leq 7$. However, if $\{a, b\} \subset V^*$ is a basis for Q, then

$$h(Q \otimes V^*) = \sum_{j=1}^{4} k\ \text{col}_j(N(a)) + \sum_{j=1}^{4} k\ \text{col}_j(N(b)),$$

which is the span of the columns of a 10×8 matrix $N(a, b)$ whose first four columns form $N(a)$ and whose last four columns form $N(b)$. Hence

$$\begin{aligned} \mathcal{L}^\perp &= \{Q \in \mathbb{G}^2(V^*) : \text{rank}(h|_{Q\otimes V^*}) \leq 7\} \\ &= \{Q = ka \oplus kb \in \mathbb{G}^2(V^*) : \text{rank}(N(a, b)) \leq 7\}. \end{aligned}$$

Writing $a = \sum_{i=1}^4 a_i x_i$ and $b = \sum_{i=1}^4 b_i x_i$, where $a_i, b_i \in k$, it follows that the entries of the matrix $N(a, b)$ are homogeneous degree-one polynomials in a_i and b_i. Any 8×8 minor of $N(a, b)$ is obtained by omitting two rows of $N(a, b)$ and computing the determinant of the resulting submatrix, so such a minor is a bihomogeneous polynomial in a_i and b_i of bidegree

(4,4). In particular, 8×8 minors of $N(a,b)$ may be expressed as homogeneous degree-four polynomials in the Plücker coordinates $N_{ij} = a_i b_j - a_j b_i$ evaluated at Q. Let $J_N < S_N$ be the ideal generated by these 45 maximal minors. We have then a global-coordinate description,

$$\mathcal{L}^{\perp} = \text{Proj}\left(\frac{S_N}{\langle p_N\rangle + J_N}\right).$$

In practice, the quartic polynomials which generate J_N are easily calculated by computer, but usually they are not enlightening geometrically.

Attempting to conduct the same analysis on the scheme $\mathcal{L}$, we first observe

$$\mathcal{L} = \{q \in \mathbb{G}_2(V) : \text{rank}(f|_{q\otimes V}) \leq 5\}.$$

If $q = ka \oplus kb \in \mathbb{G}_2(V)$, let $M(a,b)$ denote the 6×8 matrix whose first four columns form $M(a)$ and whose last four columns form $M(b)$. It follows that

$$\mathcal{L} = \{q = ka \oplus kb \in \mathbb{G}_2(V) : \text{rank}(M(a,b)) \leq 5\}.$$

Writing $a = \sum_1^4 a_i e_i$ and $b = \sum_1^4 b_i e_i$, the Plücker coordinates are $M_{ij} = a_i b_j - a_j b_i$. In general, not all of the 6×6 minors of $M(a,b)$ can be expressed as polynomials in the M_{ij}. However, certain combinations of the 6×6 minors are polynomials in the M_{ij} as follows.

Lemma 1.1. *If $1 \leq r < s \leq 8$, let $q_{rs} = q_{rs}(a,b)$ denote the 6×6 minor of $M(a,b)$ obtained by omitting columns r and s. The ten polynomials:*

$$q_{15}, \quad q_{26}, \quad q_{37}, \quad q_{48},$$
$$q_{16} + q_{25}, \quad q_{17} + q_{35}, \quad q_{18} + q_{45}, \quad q_{27} + q_{36}, \quad q_{28} + q_{46}, \quad q_{38} + q_{47},$$

are homogeneous cubic polynomials in the Plücker coordinates M_{ij}. ■

Let $J_M \subset S_M$ be the ideal generated by the ten polynomials from Lemma 1.1.

Remark 1.2. The following example demonstrates that, in general, the ideal $\langle p_M\rangle + J_M \subset S_M$ does not define the scheme $\mathcal{L}$. However, in all of the examples of quantum $\mathbb{P}^3$s we have examined, these polynomials do define $\mathcal{L}$. We verify this on a case-by-case basis. Let K be the image of J_N under the orthogonality isomorphism $S_N \to S_M$. Case by case, it is not difficult to check that $\langle p_M\rangle + K$ and $\langle p_M\rangle + J_M$ have the same saturation as graded ideals in S_M. In the subsequent sections we will omit these details.

Example 1.3. For the algebra $k[x_1, \dots, x_4]$ with defining relations

$$x_1 \otimes x_1 + x_1 \otimes x_2 = 0, \qquad x_1 \otimes x_3 = 0, \qquad x_1 \otimes x_4 = 0,$$
$$x_1 \otimes x_2 + x_2 \otimes x_2 = 0, \qquad x_2 \otimes x_3 = 0, \qquad x_2 \otimes x_4 = 0,$$

the ideal J_M is easily seen to be $\{0\}$. The scheme $\mathcal{L}^\perp$ is a proper subscheme of $\mathbb{G}^2(V^*)$, since $(N_{34})^4$ belongs to J_N. Since $\mathcal{L}^\perp \cong \mathcal{L}$, the ideal J_M does not define $\mathcal{L}$.

The subsequent sections contain several examples of line schemes of quantum $\mathbb{P}^3$s. The notation established in this section and the notation $A = T(V^*)/I$ will be in force throughout those sections. A closed subscheme of $\mathbb{P}(V)$ defined by a set R of homogeneous polynomials in the variables $x_1, \dots, x_4$ will be denoted $\mathcal{V}(R)$. Closed subschemes of $\mathbb{G}_2(V)$ will be denoted $\mathcal{V}_{\mathbb{G}}(R)$, where R is a set of homogeneous polynomials in the coordinates M_{ij}, that is, $\mathcal{V}_{\mathbb{G}}(R) = \operatorname{Proj}(S_M/\langle R, p_M\rangle)$.

2. Algebras with Symmetric Relations

In this section, we show that a quantum $\mathbb{P}^3$ with enough symmetry in its defining relations has a line scheme of dimension at least 2, with infinitely many line modules incident to any point module. This information extends [8, Corollary 2.6 and Proposition 3.1]. The result applies, in particular, to regular Clifford algebras. We consider, in some detail, the Clifford algebras studied extensively in [6]. In contrast, we also consider a non-Clifford algebra from [6] which has a 1-dimensional line scheme.

2.1. **Algebras with Symmetric Relations.** Let Sym denote the subspace of symmetric elements of $V^* \otimes V^*$ and let Λ be the subscheme of $\mathbb{P}(V^* \otimes V^*)$ of elements of rank at most two. Given $\sigma \in \operatorname{Aut}(\mathbb{P}(V^*))$, we obtain an automorphism $\bar{\sigma} :=$ identity $\otimes\, \sigma$ of $\mathbb{P}(V^* \otimes V^*)$. Let $E = T(V^*)/\langle \mathrm{Sym}\rangle$ denote the exterior algebra on V^*.

Theorem 2.1. *Suppose that $A = T(V^*)/I$ is a quantum $\mathbb{P}^3$. If, for some $\tau \in \operatorname{Aut}(\mathbb{P}(V^*))$, A has the twisted algebra E^τ as a factor algebra, then every component of the line scheme $\mathcal{L}$ has dimension at least two, and, for any $p \in \mathcal{P}$, the subscheme $\mathcal{L}_p$ of $\mathcal{L}$, consisting of the line modules which map onto the point module $M(p)$, has dimension at least one.*

Proof. Let $\sigma = \tau^{-1}$ and put $\mathrm{Sym}^\sigma = \bar{\sigma}(\mathrm{Sym})$. The twisted algebra E^τ is $T(V^*)/\langle \mathrm{Sym}^\sigma\rangle$ and so we may assume $I_2 \subset \mathrm{Sym}^\sigma$.

To prove the first claim it suffices to estimate the dimension of $\mathcal{H} = \mathbb{P}(I_2) \cap \Lambda$. Given that $\mathbb{P}(I_2) \cap \Lambda = \mathbb{P}(I_2) \cap (\mathbb{P}(\mathrm{Sym}^\sigma) \cap \Lambda)$, and that Λ is $\bar{\sigma}$-invariant, we have $(\mathbb{P}(\mathrm{Sym}^\sigma) \cap \Lambda) \cong (\mathbb{P}(\mathrm{Sym}) \cap \Lambda)$ which has dimension six and degree ten. Since $\mathbb{P}(I_2) \cong \mathbb{P}^5$ and since we are intersecting schemes inside $\mathbb{P}(\mathrm{Sym}^\sigma) \cong \mathbb{P}^9$, every component of our intersection, $\mathcal{H}$, must have dimension at least two.

Following the proof of [8, Proposition 3.1], and using $I_2 \subset \mathrm{Sym}^\sigma$, to prove the last statement it suffices to compute the dimension of $(\Lambda_p \cap \mathbb{P}(\mathrm{Sym}^\sigma)) \cap \mathbb{P}(I_2)$, where $p \in \mathcal{P}$, and $\Lambda_p = \{\zeta \in \Lambda | \ p\zeta = 0\}$. Fix $p \in \mathcal{P}$, let W be the vector space of symmetric tensors in $p^\perp \otimes V^*$ and let $W^\sigma = \bar{\sigma}(W)$. We see that $(\Lambda_p \cap \mathbb{P}(\mathrm{Sym}^\sigma)) \cap \mathbb{P}(I_2) \subset \mathbb{P}(W^\sigma + I_2)$. By definition, $\dim((p^\perp \otimes V^*) \cap I_2) = 3$. Since $I_2 \subset \mathrm{Sym}^\sigma$, we have $\dim(W^\sigma \cap I_2) = 3$. Hence, $\mathbb{P}(W^\sigma + I_2) \cong \mathbb{P}^8$. Inside this $\mathbb{P}^8$ we are intersecting $\mathbb{P}(I_2) \cong \mathbb{P}^5$ with $\Lambda_p \cap \mathbb{P}(\mathrm{Sym}^\sigma) = \bar{\sigma}(\Lambda_p \cap \mathbb{P}(\mathrm{Sym}))$, which has dimension four. Hence the intersection has dimension at least one, as required. ■

Remark 2.2. The theorem clearly covers any regular Clifford algebra or any twist thereof, as well as many other examples.

It should be noted that the scheme $\mathcal{H}$ of a generic Clifford algebra is irreducible and nonsingular and has dimension two, as evidenced in Theorem 2.3 below. However, not every Clifford algebra is a quantum $\mathbb{P}^3$, c.f. [3].

2.2. **A symmetric example.** Following [6], let $A(q) = T(V^*)/I$ be the quadratic algebra with I generated by the quadratic forms

$$x_1 \otimes x_2 - qx_2 \otimes x_1 - x_4 \otimes x_4, \qquad x_2 \otimes x_2 - qx_3 \otimes x_2,$$
$$x_1 \otimes x_3 - qx_3 \otimes x_1 - x_2 \otimes x_2, \qquad qx_2 \otimes x_4 - x_4 \otimes x_2,$$
$$x_1 \otimes x_4 - qx_4 \otimes x_1 - x_3 \otimes x_3, \qquad x_3 \otimes x_4 - qx_4 \otimes x_3,$$

where $q \in k^\times$. Let $A = A(-1)$. The algebra A is a Clifford algebra and a quantum $\mathbb{P}^3$.

Let S be the k-subalgebra of A generated by x_2, x_3 and x_4. In the language of [1], S is a quantum $\mathbb{P}^2$. The algebra A is an Ore extension of S, where $A = S[x_1, \theta; \delta]$. The point scheme of S is the triangle $\mathcal{V}(x_2x_3x_4) \subset \mathbb{P}^2$. Since A is an Ore extension of S, the module $M := A/(S_+)A$, is both a left and a right point module over A, and is the only point module of A. In particular, the point scheme, $\mathcal{P}$, of A has only one closed point and this point has multiplicity 20. In homogeneous coordinates on $\mathbb{P}(V)$, we have $M = M((1, 0, 0, 0))$.

Theorem 2.3. *The line scheme, $\mathcal{L}$, of the Clifford Algebra A is a 2-dimensional, irreducible, nonsingular subscheme of $\mathbb{G}_2(V)$. In Plücker coordinates on $\mathbb{G}_2(V)$, $\mathcal{L}$ is given by the polynomials:*

$$(M_{23})^3 - 2M_{12}M_{23}M_{24} - (M_{24})^2M_{34},$$

$$(M_{34})^3 + 2M_{13}M_{23}M_{34} + (M_{23})^2M_{24},$$

$$(M_{24})^3 + 2M_{14}M_{24}M_{34} + M_{23}(M_{34})^2,$$

$$2M_{12}M_{13}M_{14} - (M_{13})^2M_{23} - M_{12}(M_{23})^2 - M_{14}(M_{24})^2 + \\ +(M_{12})^2M_{24} - (M_{14})^2M_{34} - M_{13}(M_{34})^2.$$

Proof. The second statement is Lemma 1.1 and Remark 1.2.

Let U_{ij} denote the affine neighborhood of $\mathbb{G}_2(V)$ defined by $M_{ij} \neq 0$, $1 \leq i < j \leq 4$. Focusing on $\mathcal{L} \cap U_{23}$, a Gröbner-basis calculation yields

$$\mathcal{L} \cap U_{23} = \mathcal{V}(m_{14} + m_{12}m_{34} - m_{13}m_{24}, \\ m_{24} + 2m_{13}m_{34} + m_{34}^3, \quad -1 + 2m_{12}m_{24} + m_{24}^2m_{34}) \subset \mathbb{A}^5,$$

where $m_{ij} = M_{ij}/M_{23}$, from which it follows that $\mathcal{L} \cap U_{23}$ is a smooth, reduced surface. By symmetry, a similar result holds for $\mathcal{L} \cap U_{24}$ and $\mathcal{L} \cap U_{34}$. The complement of $\mathcal{V}_{\mathbb{G}}(M_{23}, M_{24}, M_{34})$ is $U_{23} \cup U_{24} \cup U_{34}$, and its intersection with $\mathcal{L}$ is irreducible. A calculation shows that $\mathcal{L} \cap \mathcal{V}_{\mathbb{G}}(M_{23}, M_{24}, M_{34}) = \mathcal{V}_{\mathbb{G}}(M_{12}M_{13}M_{14}, M_{23}, M_{24}, M_{34})$, which has three 1-dimensional components. However, by Theorem 2.1, every component of $\mathcal{L}$ has dimension at least two, so there are no components of $\mathcal{L}$ in the 2-dimensional complement of $U_{23} \cup U_{24} \cup U_{34}$. Hence, $\mathcal{L}$ is irreducible and has dimension two. With this information, a simple global computation yields that $\mathcal{L}$ is nonsingular. ■

It should be noted that the calculations which used $\mathcal{L}$ to prove Theorem 2.3 are straightforward in comparison with calculations involving the isomorphic scheme $\mathcal{H}$.

The incidence relations between the line modules of A and the unique point module of A may be described as follows. If $q \in \mathcal{L}$, then the unique point module, $M((1, 0, 0, 0))$, of A is a quotient of the line module $L(q)$ if and only if $(1, 0, 0, 0)$ lies on the line q; that is, if and only if

$$q \in \mathcal{L}_{(1,0,0,0)} := \mathcal{L} \cap \mathcal{V}_{\mathbb{G}}(M_{23}, M_{24}, M_{34}) \;= \mathcal{V}_{\mathbb{G}}(M_{12}M_{13}M_{14}, M_{23}, M_{24}, M_{34}),$$

as computed in Theorem 2.3. In other words, the point $(1,0,0,0)$ is the common point of intersection of the three coordinate planes $\mathcal{V}(x_2)$, $\mathcal{V}(x_3)$ and $\mathcal{V}(x_4)$ in $\mathbb{P}(V)$, so $\mathcal{L}_{(1,0,0,0)}$ is the triangle of $\mathbb{P}^1$s in $\mathbb{G}_2(V)$ corresponding to all the lines through $(1,0,0,0)$ lying in each of these coordinate planes.

2.3. **Fat Point Modules.** In order to obtain a better understanding of $\operatorname{Proj} A$ for our Clifford algebra A in §2.2, we examine the incidence relations between the line modules of A and the fat-point modules of A. In the language of [1], a *fat-point module* of A is a critical, graded A-module of Gelfand-Kirillov dimension one. Such a module represents a simple object in $\operatorname{Proj} A$, and this simple object is called a *fat point* (c.f. [1]). Since A is a Clifford algebra, we may apply the work of Le Bruyn in [3] on fat points of Clifford algebras to the fat points of A as follows.

The center of A is the polynomial ring $Z(A) := k[x_1^2, x_2^2, x_3^2, x_4^2]$. Let $Y = \operatorname{Proj} Z(A)$ denote the corresponding weighted $\mathbb{P}^3$ and, for $p \in Y$, let $I_Z(p)$ denote the corresponding maximal graded prime ideal of $Z(A)$. By [3, Propositions 8 & 9], there is a bijection between equivalence classes of fat points in $\operatorname{Proj} A$ and points in Y. For each $p \in Y$, let $F(p)$ be the corresponding fat-point module, which is determined by $\operatorname{Ann}_{Z(A)}(F(p)) = I_Z(p)$. For $p \neq (1,0,0,0)$, $F(p)$ has multiplicity two and we may assume the Hilbert series of $F(p)$ is $2/(1-t)$. The fat-point module $F((1,0,0,0))$ is the unique point module.

Let $p \in Y$ with $p \neq (1,0,0,0)$ and let $q \in \mathcal{L}$. We say that the fat point $F(p)$ lies on the line module $L(q)$ if there is a nontrivial, graded homomorphism $L(q) \to F(p)$.

If $w_0 \in \mathbb{P}(F(p)_0) \cong \mathbb{P}^1$, then $\dim(\operatorname{Ann}_{A_1}(w_0)) \geq 2$, since $\dim(F(p)_1) = 2$. It follows, as in the proof of [8, Lemma 3.3], that $\dim(\operatorname{Ann}_{A_1}(w_0)) = 2$. This gives a map $\zeta_p : \mathbb{P}(F(p)_0) \to \mathbb{G}_2(V)$ given by $w_0 \mapsto (\operatorname{Ann}_{A_1}(w_0))^{\perp}$. We denote the image of ζ_p by Φ_p.

Theorem 2.4. *Let $p \in Y$ with $p \neq (1,0,0,0)$. The scheme consisting of the line modules which are incident to the fat point $F(p)$ is $\Phi_p \cap \mathcal{L}$. The dimension of this scheme is zero and, counting multiplicity, there are six lines in this scheme.*

Proof. (Sketch) The first statement holds since, if $q \in \mathcal{L}$, then nontrivial, graded homomorphisms from $L(q)$ to $F(p)$ correspond to nonzero vectors $w \in F(p)_0$ such that $q^\perp \subset \text{Ann}_{A_1}(w)$. By the discussion above, such a containment is equality.

To prove the second statement, it suffices to show that $\zeta_p^{-1}(\Phi_p \cap \mathcal{L}) \subset \mathbb{P}(F(p)_0) \cong \mathbb{P}^1$ is finite with multiplicity six. This can be done case-by-case as follows. Choose coordinates for A and choose a specific coordinate representation of $F(p)$ with the action of A_1 on $F(p)_0$ given by 2×2 matrices $x_{i,0}$. Let (λ, μ) be homogeneous coordinates for $w \in \mathbb{P}(F(p)_0)$ and let $H(\lambda, \mu)$ be the 4×2 matrix whose ith row is $(\lambda, \mu) \cdot x_{i,0}$. Then $\sum_i a_i x_i \in \text{Ann}_{A_i}(w)$ if and only if (a_i) is in the left kernel of $H(\lambda, \mu)$. So the line in $\mathbb{P}(V)$ which represents the point $\zeta_p(w) \in \mathbb{G}_2(V)$ is the span of the two columns of $H(\lambda, \mu)$. To see if this point belongs to $\mathcal{L}$, we evaluate the polynomials which define $\mathcal{L}$, given in Theorem 2.3, at $H(\lambda, \mu)$. This yields a system of homogeneous equations in (λ, μ) describing exactly $\zeta_p^{-1}(\Phi_p \cap \mathcal{L})$. We omit the details that this system of equations, case-by-case, reduces to a single equation of degree six in λ and μ. We note that the equation is often not reduced. ■

2.4. **A non-symmetric example.** Assume $k = \bar{k}$ and let $i = \sqrt{-1}$ and let $B = A(i)$ as in §2.2. Like $A(-1)$, the algebra B is an Ore extension of a quantum $\mathbb{P}^2$, and hence is a quantum $\mathbb{P}^3$. It has a single point module, represented by the point $(1, 0, 0, 0)$ (cf. [6]). As evidenced by the following theorem, the line scheme of B is substantially different from that of A, although both A and B have the same set of lines incident to the unique point module. The proof, which we omit, is a local computation similar to the proof of Theorem 2.3.

Theorem 2.5. *The line scheme, $\mathcal{L}$, of the algebra B is one-dimensional with four components. The subscheme $\mathcal{L}_{(1,0,0,0)}$, of line modules incident to the unique point module, is $\mathcal{V}_{\mathbb{G}}(M_{12}M_{13}M_{14}, M_{23}, M_{24}, M_{34})$.* ■

We note that the three components of $\mathcal{L}_{(1,0,0,0)}$ are reduced. These components correspond to three of the four components of $\mathcal{L}$, but, conceivably, as subschemes of $\mathcal{L}$, they might not be reduced.

3. Quadric-Line Algebras and Quadric Algebras

In this section, we discuss two of the quantum $\mathbb{P}^3$s whose point scheme, $\mathcal{P}$, contains a smooth quadric hypersurface $Q \subset \mathbb{P}(V)$ ([9, 10]). There are several other types of such algebras as well as algebras associated to non-smooth quadric hypersurfaces, but the analysis of those examples is very similar to the two given here.

3.1. **Quadric Line Algebras.** Let $\alpha \in k$ with $\alpha(\alpha-1) \neq 0$ and define the algebra $A = A(\alpha)$ to be $T(V^*)/I$ where I is the ideal generated by the quadratic forms

$$\begin{array}{ll} x_1 \otimes x_2 - x_2 \otimes x_1, & x_3 \otimes x_2 - \alpha x_2 \otimes x_3 - (1-\alpha) x_1 \otimes x_4, \\ x_1 \otimes x_3 - x_3 \otimes x_1, & x_2 \otimes x_4 - x_4 \otimes x_2, , \\ x_1 \otimes x_4 - x_4 \otimes x_1, & x_3 \otimes x_4 - x_4 \otimes x_3. \end{array}$$

The point scheme of this algebra is isomorphic to the scheme $\mathcal{P}$ in $\mathbb{P}(V)$ which is the union of the smooth quadric $Q = \mathcal{V}(x_1x_4 - x_2x_3)$ and the line $L = \mathcal{V}(x_1, x_4)$. The scheme Γ is the graph of an automorphism of $\mathcal{P}$ whose action on Q is the identity.

The line scheme, $\mathcal{L}$, of A is $\mathcal{V}_{\mathbb{G}}(J_M)$ as described in §1. Lemma 1.1 produces the following ten cubic generators for J_M:

$$\begin{array}{ll} (\alpha-1)M_{12}M_{13}M_{14}, & (1-\alpha)M_{12}(M_{14}+M_{23})M_{14}, \\ (\alpha-1)M_{12}M_{24}M_{14}, & (1-\alpha)M_{34}(M_{14}+M_{23})M_{14}, \\ (\alpha-1)M_{13}M_{34}M_{14}, & (1-\alpha)M_{13}(M_{14}-M_{23})M_{14}, \\ (\alpha-1)M_{24}M_{34}M_{14}, & (1-\alpha)M_{24}(M_{14}-M_{23})M_{14}, \end{array}$$

$$(1-\alpha)(M_{14}^2 - M_{12}M_{34} - M_{13}M_{24})M_{14}, \qquad (*)$$

$$(1-\alpha)(M_{13}M_{14}M_{24} - M_{13}M_{23}M_{24} + M_{12}M_{14}M_{34} + M_{12}M_{23}M_{34}). \qquad (\dagger)$$

Adding $(*)$ and $(\dagger)$ and using p_M to substitute $M_{14}M_{23}$ for $M_{13}M_{24} - M_{12}M_{34}$ yields

$$(1-\alpha)(M_{14}-M_{23})(M_{14}+M_{23})M_{14},$$

with which $(\dagger)$ may be replaced. Let

$$K = \langle M_{12}, M_{34}, M_{14}-M_{23}, p_M\rangle \cap \langle M_{13}, M_{24}, M_{14}+M_{23}, p_M\rangle \cap \langle M_{14}, p_M\rangle \subset S_M.$$

It is easy to establish that $(*)$ belongs to K, from which it follows that $\langle p_M\rangle + J_M = K$. Thus,

$$\mathcal{L} = \mathcal{V}_{\mathbb{G}}(M_{12}, M_{34}, M_{14}-M_{23}) \cup \mathcal{V}_{\mathbb{G}}(M_{13}, M_{24}, M_{14}+M_{23}) \cup \mathcal{V}_{\mathbb{G}}(M_{14}),$$

so the line scheme $\mathcal{L}$ is reduced and has three components. Two of the components have dimension one and correspond to the two rulings on Q. The third component, $\mathcal{V}_{\mathbb{G}}(M_{14})$, has dimension three and corresponds to the lines in $\mathbb{P}(V)$ which meet L. To this component is associated a skew-symmetric bilinear form $b = x_1 \wedge x_4 : V \times V \to k$ such that the radical of b is the affine cone in V over L. In fact, the 2-dimensional isotropic subspaces of b in V correspond to the isomorphism classes of line modules $M(\ell)$ such that the line $\ell \not\subset Q$.

3.2. **Quadric Algebras.** In this subsection, we discuss those quantum $\mathbb{P}^3$s whose point scheme is isomorphic to a smooth quadric $Q = \mathcal{V}(x_1x_4 - x_2x_3)$ in $\mathbb{P}(V)$ ([9, 10]). By [9], such an algebra depends on $\alpha \in k$ with $\alpha(\alpha^2+1) \neq 0$ and on certain $\tau \in \mathrm{Aut}(Q)$. Let $\sigma = \tau^{-1}$. Then $A = A(\tau, \alpha) = T(V^*)/I$, where I is the ideal generated by the quadratic forms

$$a := x_1 \otimes x_4^\sigma - x_2 \otimes x_3^\sigma, \qquad b := x_3 \otimes x_4^\sigma - x_4 \otimes x_3^\sigma - x_1 \otimes x_2^\sigma + x_2 \otimes x_1^\sigma,$$
$$c := x_1 \otimes x_3^\sigma - x_3 \otimes x_1^\sigma, \qquad d := x_1 \otimes x_4^\sigma - x_4 \otimes x_1^\sigma + (1/\alpha)(x_1 \otimes x_2^\sigma - x_2 \otimes x_1^\sigma),$$
$$e := x_2 \otimes x_4^\sigma - x_4 \otimes x_2^\sigma, \qquad f := x_2 \otimes x_3^\sigma - x_3 \otimes x_2^\sigma + (1/\alpha)(x_1 \otimes x_2^\sigma - x_2 \otimes x_1^\sigma).$$

The point scheme $\mathcal{P}$ of this algebra is the quadric Q, and Γ is the graph of τ. We note that $\tau^2 \in \mathrm{Aut}(A)$.

The computation of the line scheme of A is almost identical to that of the previous subsection; the polynomial $\alpha M_{12} - M_{23} + \alpha M_{34} - M_{14}$ plays the role here that M_{14} played in the previous subsection, that is,

$$\mathcal{L} = \mathcal{V}_{\mathbb{G}}(M_{12}, M_{34}, M_{14} - M_{23}) \cup \mathcal{V}_{\mathbb{G}}(M_{13}, M_{24}, M_{14} + M_{23}) \cup \mathcal{V}_{\mathbb{G}}(\alpha M_{12} - M_{23} + \alpha M_{34} - M_{14}).$$

So the line scheme $\mathcal{L}$ is reduced and has three components. Two components have dimension one and correspond to the two rulings on Q. The third, $\mathcal{V}_{\mathbb{G}}(\alpha M_{12} - M_{23} + \alpha M_{34} - M_{14})$, has dimension three and is associated to a nondegenerate skew-symmetric bilinear form

$$B = \alpha x_1 \wedge x_2 - x_2 \wedge x_3 + \alpha x_3 \wedge x_4 - x_1 \wedge x_4 : V \times V \to k.$$

The 2-dimensional isotropic subspaces of B in V correspond to the isomorphism classes of line modules $M(\ell)$, such that $\ell \not\subset Q$.

It should be noted that, unlike the preceding example, the line scheme $\mathcal{L}$ parametrizes only the right line modules and not the left line modules. In fact, the scheme parametrizing the left line modules is

$$\mathcal{L}_{\text{Left}} = \mathcal{V}_{\mathbb{G}}(M_{12}, M_{34}, M_{14} - M_{23}) \ \cup \ \mathcal{V}_{\mathbb{G}}(M_{13}, M_{24}, M_{14} + M_{23}) \cup$$
$$\cup \ \mathcal{V}_{\mathbb{G}}(M_{12} + \alpha M_{23} + M_{34} + \alpha M_{14}).$$

Let $J = \{g \in V^* \otimes V^* : g|_{\Gamma_2} = 0\}$. By [9], $J = I_2 \oplus k\tilde{\Omega}$, where $\tilde{\Omega} = x_2^\tau \otimes x_1 - x_1^\tau \otimes x_2$. The image, Ω, of $\tilde{\Omega}$ in A is normal in A. As in [8], we define Ψ to be the subscheme of $\mathbb{P}(V \otimes V)$ formed by the union of the elements of Δ_2; that is, $\Psi = \Pi_2(\Pi_1^{-1}(\Delta_2))$, where Π_1 and Π_2 are the first and second projections, respectively, from the incidence relation $\Sigma = \{(\omega, U) \in \Delta_2 \times \mathbb{P}(V \otimes V) : U \subset \omega\}$.

Lemma 3.1. *For this algebra, we have*

(a) $I_2 = \{g \in V^* \otimes V^* : g|_{\Gamma_2} = 0$ *and* $g|_\Psi = 0\}$, *and*

(b) $I_2^\perp = \{g \in V \otimes V : g|_{\mathcal{H}} = 0\}$.

Proof. (a) The line $\ell = \mathcal{V}(x_3, x_4 - \alpha x_2) = \mathbb{P}(ke_1 \oplus k(e_2 + \alpha e_4))$ in $\mathbb{P}(V)$ corresponds to a right line module of A, that is, a point in $\mathcal{L}$. The image of ℓ in Δ_2 is the 3-dimensional subspace $\ell \otimes V \cap I_2^\perp$ of $V \otimes V$, which contains the 1-dimensional space $U = k(e_1 \otimes (e_4 - \alpha e_2)^\tau + (e_2 + \alpha e_4) \otimes e_3^\tau)$. So U is a closed point of Ψ. Since $\tilde{\Omega}$ does not vanish on U, (a) follows.

(b) Each of the quadratic forms a, c, d, e and f has rank two. The form b may be replaced in the basis for I_2 by the form

$$b + e - c = (x_2 + x_3) \otimes (x_1 + x_4)^\sigma - (x_1 + x_4) \otimes (x_2 + x_3)^\sigma,$$

which also has rank two. Thus, I_2 has a basis of rank-two elements and these rank-two elements all represent closed points in $\mathcal{H}$. Hence, (b) follows. ■

The preceding lemma completes Example 4.5 of [8].

4. Homogenized $\mathfrak{sl}(2)$

Throughout this section, we assume $\text{char}(k) = 0$.

Let A denote the standard homogenization of the universal enveloping algebra of the Lie algebra $\mathfrak{sl}(2)$. The algebra A is a quantum $\mathbb{P}^3$ and has been discussed in [4, 5]. The *variety* of line modules over A was parametrized in [4] in terms of certain quadrics in $\mathbb{P}^3$. In this section, we describe the line scheme of A as a subscheme of $\mathbb{G}_2(V)$. It is neither irreducible nor reduced. It has two components, namely, a plane counted with multiplicity four and a reduced "cylinder".

We present A as $A = T(V^*)/I$, where I is the ideal generated by the quadratic forms

$$\begin{array}{ll} x_1 \otimes x_4 - x_4 \otimes x_1, & x_1 \otimes x_2 - x_2 \otimes x_1 - x_3 \otimes x_4, \\ x_2 \otimes x_4 - x_4 \otimes x_2, & x_3 \otimes x_1 - x_1 \otimes x_3 - 2x_1 \otimes x_4, \\ x_3 \otimes x_4 - x_4 \otimes x_3, & x_3 \otimes x_2 - x_2 \otimes x_3 + 2x_2 \otimes x_4. \end{array}$$

Viewed in $\mathbb{P}(V)$, the point scheme $\mathcal{P}$ of A is the union of the isolated point $e_4 = (0, 0, 0, 1)$, the plane $\mathcal{V}(x_4)$ and the embedded conic $C := \mathcal{V}(x_4, 4x_1x_2 + x_3^2)$ counted with multiplicity two ([4]).

The line scheme, $\mathcal{L}$, of A may be computed from Lemma 1.1 and Remark 1.2 as $\mathcal{V}_{\mathbb{G}}(J_M)$ where $\langle p_M \rangle + J_M$ is generated by

$$\begin{array}{lll} M_{14}(4M_{14}M_{24} + (M_{34})^2), & M_{14}(2M_{12}M_{14} + M_{13}M_{34}), & 4M_{13}(M_{24})^2 - M_{23}(M_{34})^2, \\ M_{24}(4M_{14}M_{24} + (M_{34})^2), & M_{24}(2M_{12}M_{24} - M_{23}M_{34}), & 4M_{23}(M_{14})^2 - M_{13}(M_{34})^2, \\ M_{34}(4M_{14}M_{24} + (M_{34})^2), & M_{34}(M_{14}M_{23} + M_{13}M_{24}), & p_M. \end{array}$$

It should be noted that swapping subscripts 1 and 2 yields an automorphism of S_M which leaves invariant the ideal $\langle p_M \rangle + J_M$. Moreover, $(M_{i4})^3((M_{12})^2 - M_{13}M_{23}) \in \langle p_M \rangle + J_M$, where $i = 1, 2$, since, for $\{i, j\} = \{1, 2\}$, we have

$$\begin{aligned} 4(M_{i4})^3((M_{12})^2 - M_{13}M_{23}) &= M_{i4}(2M_{12}M_{i4} - M_{i3}M_{34})(2M_{12}M_{i4} + M_{i3}M_{34}) + \\ &\quad - M_{i4}M_{i3}(4M_{j3}(M_{i4})^2 - M_{i3}(M_{34})^2), \end{aligned}$$

and the right-hand side belongs to $\langle p_M \rangle + J_M$.

The following result is proved by using the above generators and analyzing the line scheme $\mathcal{L}$ on the affine open sets of $\mathbb{G}_2(V)$.

Lemma 4.1.

(a) *The lines on the plane $\mathcal{V}(x_4)$ correspond to line modules.*

(b) *Any line in $\mathbb{P}^3$ that corresponds to a line module intersects the conic C.*

(c) *Any line in $\mathbb{P}^3$ that corresponds to a line module and which passes through a point $p = (p_1, p_2, p_3, 0) \in C$ either lies on the plane $\mathcal{V}(x_4)$ or lies on the plane $T_p = \mathcal{V}(2p_2x_1 + 2p_1x_2 + p_3x_3)$.*

(d) *For all $p \in C$, the lines on T_p which pass through p correspond to line modules.*

(e) *For all $p \in C$, the plane T_p meets the rank-3 quadric $\mathcal{V}(x_3^2 + 4x_1x_2)$ in a double line, and is the plane spanned by the point e_4 (corresponding to the unique isolated point module) and the tangent line to C on $\mathcal{V}(x_4)$ at p.* ■

Lemma 4.2. *The reduced scheme of the line scheme of A is the union of the plane $X = \mathcal{V}(M_{14}, M_{24}, M_{34})$ and the "cylinder" Y determined by the polynomials*

$$2M_{12}M_{14} + M_{13}M_{34}, \qquad 2M_{12}M_{24} - M_{23}M_{34}, \qquad M_{14}M_{23} + M_{13}M_{24},$$

$$p_M, \qquad 4M_{14}M_{24} + (M_{34})^2, \qquad (M_{12})^2 - M_{13}M_{23}.$$

Proof. The first component is given by Lemma 4.1. By analyzing the affine open sets of $\mathbb{G}_2(V)$, it is straightforward to prove that Y parametrizes those lines ℓ in $\mathbb{P}^3$ with the property that $\ell \cap C = \{p\}$ and $\ell \subset T_p$. ■

Theorem 4.3. *The line scheme of A consists of two components:*
- *the plane $X = \mathcal{V}(M_{14}, M_{24}, M_{34})$ counted with multiplicity four, and*
- *the "cylinder" Y given in Lemma 4.2 counted with multiplicity one.*

Proof. By Lemma 4.2, X and Y determine associated prime ideals of $\langle p_M \rangle + J_M$. For each i, j with $i \neq j$, let U_{ij} denote the affine open set of $\mathbb{G}_2(V)$ defined by $M_{ij} \neq 0$ and let m_{ab} denote M_{ab}/M_{ij}.

To compute the multiplicity of X, we localize $\langle p_M \rangle + J_m$ at $\langle M_{14}, M_{24}, M_{34} \rangle$ on U_{12}, U_{13} and U_{23} in turn. On U_{12}, the local ring has a subring $F = \mathbb{C}(m_{13}, m_{23})$, which is a field. As a vector space over F, the local ring has a basis $\{1, m_{14}, m_{24}, (m_{14})^2\}$, where $(m_{14})^3 = 0$ and $F(m_{14})^2 = F(m_{24})^2 = Fm_{14}m_{24}$. Hence, the ideal of longest length is the local ring itself, with length four. A similar argument holds on U_{13} and on U_{23}, and is left to the reader. It follows that X has multiplicity four.

Computing the multiplicity of Y on any U_{ij} entails inverting all the M_{ij}. It is clear that the images of the above first five generators of the ideal giving the scheme Y are zero in the local ring. The remaining generator, $(M_{12})^2 - M_{13}M_{23}$, also has image which is zero, since (as given above) $(M_{i4})^3((M_{12})^2 - M_{13}M_{23}) \in \langle p_M \rangle + J_M$, where $i = 1, 2$, and, in the local ring, m_{i4} is invertible. Hence, on each of the affine open sets, the local ring is a field, and so is an ideal of length one. Thus, the multiplicity of Y is one.

By Lemma 4.2, it remains to prove that the line scheme of A has no embedded components. This would follow if the line scheme were shown to be equidimensional. Since A is a quantum

$\mathbb{P}^3$, the line scheme of A is isomorphic to the subscheme $\mathcal{H}$ of $\mathbb{P}(V^* \otimes V^*)$ defined in §1, so it suffices to prove that $\mathcal{H}$ is equidimensional.

Let Λ denote the irreducible subscheme of $\mathbb{P}(V^* \otimes V^*)$ of elements of rank at most two. We have that $\mathcal{H} = \mathbb{P}(I_2) \cap \Lambda$. Using $\{e_{ij} : 1 \leq i, j \leq 4\}$ as homogeneous coordinates on $\mathbb{P}(V^* \otimes V^*)$, define

$$\Phi := \mathcal{V}(e_{12} + e_{21}, e_{13} + e_{31}, e_{32} + e_{23}, e_{11}, e_{22}, e_{33}).$$

The scheme $\mathbb{P}(I_2)$ is a linear subscheme of Φ, as $\mathbb{P}(I_2) = \Phi \cap \mathcal{V}(e_{44}, e_{14} + e_{41} + 2e_{31}, e_{24} + e_{42} + 2e_{23}, e_{34} + e_{43} + e_{12})$. The scheme Φ is isomorphic to $\mathbb{P}^9$, and $\mathcal{H} = \mathbb{P}(I_2) \cap \Lambda = \mathbb{P}(I_2) \cap (\Lambda \cap \Phi)$, as $\mathbb{P}(I_2) \subset \Phi$. Since the dimension of the latter intersection may be computed in the linear scheme Φ, the irreducible components of $\mathcal{H}$ have dimension at least $5 + \dim(\Lambda \cap \Phi) - 9$. By Lemma 4.2, the irreducible components of $\mathcal{H}$ have dimension at most two, so the result would follow if $\dim(\Lambda \cap \Phi) = 6$.

Let D_{ij} denote $\{p \in \Phi : e_{ij}(p) \neq 0\}$, which is an open subset of Φ. The set $\Lambda \cap D_{12}$ is contained in

$$\mathcal{V}(e_{13}e_{24} - e_{14}e_{23} - e_{12}e_{34}, \quad e_{13}e_{42} - e_{41}e_{23} - e_{12}e_{43}, \quad e_{14}e_{42} - e_{24}e_{41} - e_{12}e_{44}),$$

so there are exactly six free coordinates on the affine open set $\Lambda \cap D_{12}$. It follows that $\Lambda \cap D_{12}$ is irreducible of dimension six. By the symmetry of the defining relations of Φ, the same holds for $\Lambda \cap D_{13}$ and for $\Lambda \cap D_{23}$. Moreover, $\Lambda \cap \Phi$ contains $\mathcal{V}(e_{12}, e_{13}, e_{23}) \cap \Phi$ (the complement in $\Lambda \cap \Phi$ of $\Lambda \cap (D_{12} \cup D_{13} \cup D_{23})$), which is irreducible of dimension six. Hence, every irreducible component of $\Lambda \cap \Phi$ has dimension six, which completes the proof. ■

The proof of the preceding result highlights that coordinate computations are more easily done with the scheme $\mathcal{L}$, but dimension-counting arguments are more easily done with the scheme $\mathcal{H}$, so both descriptions, $\mathcal{L}$ and $\mathcal{H}$, of the line scheme of a quantum $\mathbb{P}^3$ can be useful in analyzing the line scheme.

Remark 4.4. For each $p \in C$, the plane T_p is determined by p and the Casimir element c of $U(\mathfrak{sl}(2))$ as follows. The Casimir element is the 2-tensor $c = 2x_2 \otimes x_1 + 2x_1 \otimes x_2 + x_3 \otimes x_3$, and we may evaluate c at any point $p = (p_i) \in \mathbb{P}(V)$ in either the left or right component to obtain the 1-tensor $c_p = 2p_2x_1 + 2p_1x_2 + p_3x_3$. Hence, if $p \in C$, then $T_p = \mathcal{V}(c_p)$. As discussed below, this description of T_p can be used to relate line modules over A to Verma modules over $U(\mathfrak{sl}(2))$. Note that the center of A is $k[\bar{c}, x_4]$, where $\bar{c}$ denotes the image of c in A (c.f., [4]).

Suppose that $\ell \subset \mathbb{P}(V)$ is a line corresponding to a point of $Y \setminus X$ in $\mathbb{G}_2(V)$. Let $p = (p_1, p_2, p_3, 0)$ denote $\ell \cap C$. If $p_3 \neq 0$, then ℓ may be written as $\ell = T_p \cap \mathcal{V}(p_2 x_1 - p_1 x_2 - \lambda x_4)$ for some $\lambda \in k$, whereas if $p_3 = 0$, then $\ell = T_p \cap \mathcal{V}(x_3 - \lambda x_4)$ for some $\lambda \in k$. In either case, $\ell = \mathcal{V}(c_p, H - \lambda x_4)$, for some $\lambda \in k$, where $c_p, H \in A_1$ span a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{sl}(2)$, where $\mathfrak{sl}(2)$ is viewed as a subspace of A_1. It follows that the line module over A corresponding to ℓ is the standard homogenization of the Verma module, $V_\lambda = U(\mathfrak{sl}(2)) \otimes_{U(\mathfrak{b})} k_\lambda$, of highest weight λ, where k_λ denotes the 1-dimensional $\mathfrak{b}$-module such that H acts by $\lambda \in k$ and c_p acts by zero.

Finally, we note that lines on $\mathcal{V}(x_4)$ correspond to line modules which are lifts of the line modules over A/Ax_4, which is isomorphic to the polynomial ring on three variables.

Remark 4.5. If ℓ is a line in $\mathbb{P}^3$ corresponding to a point of Y, then ℓ corresponds to a 2-dimensional isotropic subspace of a symmetric degenerate bilinear form $b = 2(e_1 \otimes e_2 + e_2 \otimes e_1) + e_3 \otimes e_3$, whose image in $\mathbb{P}^3$ meets C at $C \cap \ell$. All the 2-dimensional isotropic subspaces of b may be described by such lines ℓ. The radical of b is ke_4, the point in $\mathbb{P}^3$ giving the unique isolated point module.

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