The Interplay of Algebra and Geometry in the Setting of AS-Regular Algebras

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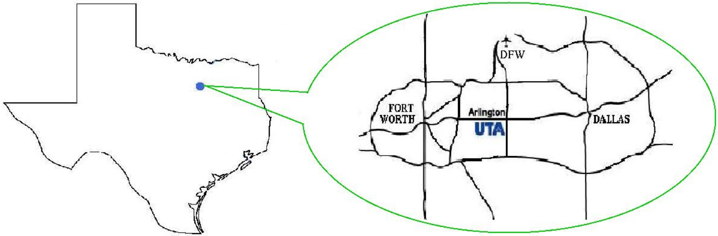

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Fix a field $\mathbb{k}$.
Consider the associative $\mathbb{k}$-algebra, S, on generators $z_1, \ldots, z_n$ with defining relations:

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ATV's idea: use certain modules (representations) in place of points/lines/planes etc....

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Similarly, for other d-linear modules & truncated d-linear modules.

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Last condition is a symmetry property & replaces commutativity.

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- For $\text{gldim}(A) = 3$, have 9 generic quadratic types & some with cubic relations. Generic types classified by ATV based on the point scheme X.

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Can such algebras be generalized?

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Example ($n = 2$)

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Assumption

For the rest of the talk, assume $\mu_{ii} = 1 \ \ \forall\ i$ (& $\Bbbk$ still alg closed).

Graded Skew Clifford Algebras

Definition (char($\mathbb{k}$) $\neq 2$ [Van den Bergh, Le Bruyn])

Let $M_1, \ldots, M_n \in M(n, \mathbb{k})$ denote symmetric matrices. A *graded Clifford algebra*, associated to $M_1, \ldots, M_n$, is a graded $\mathbb{k}$-algebra on degree-1 generators $x_1, \ldots, x_n$ and on degree-2 generators $y_1, \ldots, y_n$ with defining relations given by:

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Definition ($\text{char}(\Bbbk) \neq 2$ [Cassidy, Vancliff])

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Graded Skew Clifford Algebras

Definition (char($\mathbb{k}$) $\neq 2$ [Cassidy, Vancliff])

With μ as above, let $M_1, \ldots, M_n \in M(n, \mathbb{k})$ denote μ-symmetric matrices. A *graded skew Clifford algebra*, associated to μ, $M_1, \ldots, M_n$, is a graded $\mathbb{k}$-algebra A on degree-1 generators $x_1, \ldots, x_n$ and on degree-2 generators $y_1, \ldots, y_n$ with defining relations given by:

(i) $x_i x_j + \mu_{ij} x_j x_i = \sum_{k=1}^{n} (M_k)_{ij} y_k$ for all $i, j = 1, \ldots, n$, and

(ii) the existence of a normalizing sequence $\{y_1', \ldots, y_n'\} \subset A_2$ that spans $\mathbb{k}y_1 + \cdots + \mathbb{k}y_n$.

Example

Skew polynomial rings on generators $x_1, \ldots, x_n$ with relations $x_i x_j = -\mu_{ij} x_j x_i$, for all $i \neq j$, are GSCAs.

Example ($n = 2$: quantum affine plane)

Let $M_1 = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}.$

$$2x_1^2 = 2y_1, \quad 2x_2^2 = 2y_2,$$
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Example

The quadratic AS-reg algebra found by Shelton & Tingey in 2001 that has gldim 4 & exactly 20 nonisom point mods and a 1-dimensional line scheme is a GSCA.

Remarks

- $x_j x_i + \mu_{ji} x_i x_j =$
$$\sum_{k=1}^{n} (M_k)_{ji}\, y_k \quad \text{for all } i, j = 1, \ldots, n.$$

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It remains to generalize notions of quadatic form and quadric to try to relate properties of GSCA to some geometry.

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E.g., $(z_j z_i - \mu_{ij} z_i z_j)(\ (a_1, \ldots, a_n),\ (b_1, \ldots, b_n)\ ) = a_j b_i - \mu_{ij} a_i b_j \in \{0, 1\}$.

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If $\mu_{12} = -1$, this is the usual Jordan algebra/plane.

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Theorem ([NVZ] char($\mathbb{k}$) $\neq 2$)

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In [AS, ATV1], such algebras are classified into types A, B, E, H,

where some members of each type might not have an elliptic curve as their point scheme, but a generic member does.

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In (iii), $a^3 \neq b^3 \neq c^3 \neq a^3$ is still open.

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8. [Stephenson, V] $\Rightarrow \exists$ quad AS-reg algs of gldim 4 with 2 nonisom point mods; what about 19?

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Conclusion

There are many open problems in this rich subject, and some of them are very accessible to junior researchers.

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