THE POINTS OF QUADRATIC ALGEBRAS

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INTRODUCTION

Artin-Schelter regular algs = non-comm version of poly algs. E.g., many quantum groups, Sklyanin algebras, etc.
AS-regular algs of gldim 3 are classified (Artin, Schelter, Tate, Van den Bergh (deg 1 gens) & Stephenson (non-deg-1 gens)).

Approach of classification led to idea of point modules, line modules, etc. Useful tool = scheme which represents the functor of point modules (this scheme later called "point scheme").

Classification of AS-regular algs of gldim 4 ??
Approach = geometric. Do geometry with graded modules.

$k =$ alg. closed field, char$(k) \neq 2$.

DEFINITIONS

[AS] A connected, positively graded k-algebra A which is gen by deg-1 elements is called Artin-Schelter regular of dimension d if

(a) A has global dimension $d < \infty$,

(b) A has poly growth, and

(c) $\text{Ext}^i_A(k, A) = \delta^i_d k[\text{shift}]$. (Gorenstein cond'n)

((c) = symmetry on resolution of k.)

[Lev] Auslander-regular: (b)&(c) $\leftrightarrow$ for every fin gen A-module M, for every $i \geq 0$ & for every A-submodule N of $\text{Ext}^i_A(M, A)$, we have $\inf\{j : \text{Ext}^j_B(N, B) \neq 0\} \geq i$.

[Lev] Aus-reg + poly growth $\Rightarrow$ AS-reg.

[ATV] A as above. A point module (resp., line module) over A is a cyclic, graded A-module with Hilbert series $(1-t)^{-1}$ (resp., $(1-t)^{-2}$).

EXAMPLE

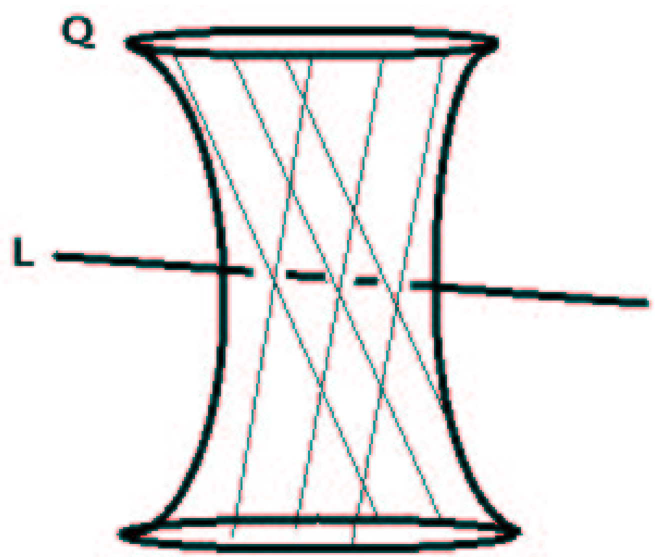
Coordinate Ring of Quantum 2×2 matrices is the k-algebra on a, b, c, d satisfying

$$\begin{aligned} ab &= qba, & bd &= qdb, \\ ac &= qca, & cd &= qdc, \\ bc &= cb, & ad - da &= (q - q^{-1})bc, \end{aligned}$$

where $q \in k^\times$ [FRT]. It is AS-reg and Aus-reg of gldim 4. The zero locus

$\{ \ p \in \mathbb{P}^3 \times \mathbb{P}^3 : f|_p = 0 \ \forall \ f \in$ span(def relns) $\}$

of the def relns is the graph of an automorphism of a quadric Q union a line L in $\mathbb{P}^3$, such that $Q \cap L = 2$ points.



The space of deg-2 forms that vanish on the zero locus of the def relns is the span of the def relns.

zero locus $\cong$ point scheme.

There is a 3-parameter family of line modules.

EXAMPLE [Vancliff Van Rompay Willaert]

The k-algebra on $x_1, \ldots, x_4$ with defining relations

$$
\begin{aligned}
x_1x_2 - qx_2x_1 &= x_4^2, & x_2x_3 &= qx_3x_2,\\
x_1x_3 - qx_3x_1 &= x_2^2, & x_3x_4 &= qx_4x_3,\\
x_1x_4 - qx_4x_1 &= x_3^2, & x_4x_2 &= qx_2x_4,
\end{aligned}
$$

where $q \in k$, $q^4 = 1$, is AS-reg and Aus-reg. It is an iterated Ore-extension (hence, noetherian, Hilbert series $= (1-t)^{-4}$, etc.).

- If $q \neq 1$, then it has one point module. The point scheme is isomorphic to the zero locus in $\mathbb{P}^3 \times \mathbb{P}^3$ of the defining relations and consists of one point of multiplicity 20.

- If $q = -1$, then there is a 2-parameter family of line modules, whilst if $q = \pm\sqrt{-1}$, then there is a 1-parameter family of line modules.

POINTS

My interest = quadratic algs, 4 gens, 6 relns AS-reg (or Aus-reg).... to be "non-comm version" of poly alg on 4 gens. Try to construct such alg from geometric data.

LEMMA [Van den Bergh] If A is a quadratic algebra such that $A = \dfrac{k\langle x_0, x_1, x_2, x_3\rangle}{\langle \text{6 generic relations}\rangle}$, then the zero locus,

$$\Gamma = \{p \in \mathbb{P}^3 \times \mathbb{P}^3 : f|_p = 0 \ \forall \ f \in \text{ def relns}\},$$

of the defining relns, is finite, and consists of 20 points (counted with mult.).

NOTE If A is quadratic, Aus-reg, noetherian, $H(t) = (1-t)^{-4}$, then $\Gamma \cong$ scheme that reps functor of point mods [Vancliff Van Rompay] [Shelton Vancliff].

PROOF

$$
\begin{array}{rcll}
\Gamma \subset \mathbb{P}^3 \times \mathbb{P}^3 & \hookrightarrow & \mathbb{P}^{15} & \text{(Segre)} \\
(u,v) & \mapsto & uv^T & \mathbb{P}(\text{rk-one } 4\times 4 \text{ matrices}) \\
\text{free alg level} \quad x_i x_j & \mapsto & x_{ij} & ij\text{-coord function} \\
\text{deg-2 relns} & \mapsto & \text{deg-1 polys in } x_{ij} \text{ on } \mathbb{P}^{15} &
\end{array}
$$

$$
\Rightarrow \ \Gamma \cong \underbrace{\mathcal{V}(6 \text{ generic deg-1 polys})}_{\text{generic } \mathbb{P}^9 \text{ in } \mathbb{P}^{15}} \cap \underbrace{\mathbb{P}(\text{rk-one } 4\times 4 \text{ matrices})}_{\text{dim 6, deg 20}}
$$

$\Rightarrow \Gamma \cong$ scheme of dim $= (6+9) - 15 = 0$,

of deg 20 (Bertini)

$\Rightarrow \Gamma =$ scheme of 20 points (counted with mult.)

Moreover, $\exists$ family whose generic member has 20 distinct points (& is AS-reg & Aus-reg too). ∎

Without knowing the proof of this lemma, it is at least believable.

Can defining relns of A be recovered from Γ?

It is less believable that, when Γ is finite,

span of def relns of A could equal $\{$ deg-2 forms that vanish on $\Gamma\}$.

[Shelton V]

earlier quadratic algebra on 4 gens with 6 relns where $|\Gamma| < \infty$ (one point of mult 20) shown to satisfy $=$.

[V Van Rompay] [Shelton V]

construct AS-reg (& Aus-reg) quadratic algs on 4 gens with 6 def relns where $|\Gamma| = \infty$

($\Gamma \cong$ quadric in $\mathbb{P}^3$) where have $\subset$.

This seems backwards! Accident??

THEOREM [Shelton V]

Let A denote a quadratic algebra,

$A = \dfrac{k\langle x_0, x_1, x_2, x_3\rangle}{\langle 6 \text{ relations}\rangle}$, and let Γ denote the zero locus in $\mathbb{P}^3 \times \mathbb{P}^3$ of the defining relns of A.

If Γ is **finite**, then

$$\text{span(relns of } A) = \{f \in k\langle x_0, \ldots, x_3\rangle_2 : f|_\Gamma = 0\}.$$

NOTE

No hypothesis on reg, noeth, Hilbert series, etc.

PROOF $\Gamma \hookrightarrow$ scheme $X = \mathbb{P}$(rk-one 4×4 matrices) = image of $\mathbb{P}^3 \times \mathbb{P}^3$.

X = Cohen-Macaulay scheme.

deg-2 relns $\mapsto$ deg-1 polys in the x_{ij} viewed in homog coord ring R of X.

Let $I = \langle$the 6 deg-1 polys$\rangle \subset R$.

$\mathcal{V}_X(I)$ = image of $\Gamma \subset X$.

$\dim X = 6 = 6 + 0 = \#$ deg-1 polys $+ \dim(\text{im } \Gamma)$.

So, we may apply Macaulay's Unmixedness Theorem to R and I ...

> In a Cohen-Macaulay ring, if I is an ideal gen by n elements such that $\text{codim} I = n$, then every assoc prime of I is min over I.

In our case,

$\text{codim} I = \dim X - \dim \mathcal{V}_X(I) = 6 = \#$ gens of I,

so every assoc prime of I is min over I.

If result were false, then there is a deg-2 $f \notin$ span of def relns of A such that $f|_\Gamma = 0$.

$f \mapsto$ deg-1 poly, F, on X, $F \in \text{sat}(I) \setminus I \subset R$.

$\Rightarrow x_{ij}^n F \in I$ for some $n \in \mathbb{N}$ $\forall$ i, j.

$\Rightarrow$ irrel ideal $\mathfrak{m} =$ ann of nonzero element of R/I.

$\Rightarrow \mathfrak{m} =$ assoc prime of I.

$\mathfrak{m}$ not min over $I \Rightarrow$ supposition false. ■

COROLLARY [Shelton V]

If A is a quadratic algebra such that

$$A = \frac{k\langle x_0, x_1, x_2, x_3 \rangle}{\langle \text{6 generic relations} \rangle}, \qquad \text{then}$$

span(relns of A) $= \{f \in k\langle x_0, \ldots, x_3 \rangle_2 : f|_\Gamma = 0\}$. ■

LINES

To generalise Γ, would like to think of Γ as point scheme, which it is if A quadratic, Aus-reg of gldim 4, noetherian, etc. [VV], [SV] (both Γ & point scheme exist without reg hyps [ATV]).

LEMMA [SV] If A is a positively graded, conn. k-algebra gen by deg-1 elements, then there is a scheme that reps the functor of line mods (in fact, d-linear mods, where $d = 0 \leftrightarrow$ point, $d = 1 \leftrightarrow$ line, etc). ■

Call this scheme the "line scheme" of A.

If we also assume A quadratic, 4 gens, 6 relns, Aus-reg, etc, then may view line scheme in different ways.

Sensible way: subscheme of Grassmannian of lines in $\mathbb{P}^3$.

Less sensible way: line scheme $\mathcal{L} \cong$ scheme of rank ≤ 2 elements in $\mathbb{P}$(span def relns of A).

THEOREM [Shelton V]

Let A be quadratic, noetherian, Aus-reg alg of gldim 4 such that $H(t) = (1-t)^{-4}$, and write

$$A^! = \frac{k\langle e_0, \ldots, e_3\rangle}{\langle\{\text{span of def relns of } A\}^\perp\rangle}.$$

If dim(line scheme $\mathcal{L}$) $= 1$, then

span(relns of A) $= \{g \in k\langle e_0, \ldots, e_3\rangle_2 : g|_{\mathcal{L}} = 0\}^\perp$.

PROOF

$\mathbb{P}(k\langle x_0,\ldots,x_3\rangle_2) = \mathbb{P}^{15} = \mathbb{P}(4\times 4$ matrices$)$.

$e_ie_j \mapsto e_{ij}$. $Y = \mathbb{P}($ rank≤ 2 elements $) \subset \mathbb{P}^{15}$.

$Y =$ Cohen-Macaulay scheme of dim 11.

$\mathcal{L} \cong$ subscheme $\mathcal{V}_Y(J)$ of Y where $J \subset$ homog coord ring of Y and J gen by 10 deg-1 polys in the e_{ij} determined by $\{$ def relns of A $\}^{\perp}$.

$\dim Y = 11$

$$= 10 + 1 = \#\text{ deg-1 polys } + \dim(\text{im } \mathcal{L}).$$

Apply Macaulay's Unmixedness Theorem. ■

NOTE: $\dim(\mathcal{L}) = 1$ is minimal since $\dim = 11 + 5 - 15 = 1$ [VdB].

Examples of Stafford modelled on Sklyanin algebra of gldim 4 have infinite point scheme (i.e., $|\Gamma| = \infty$) but 1-diml line scheme, so theorem applies.

Both theorems essentially say that for quadratic algebras on 4 gens with 6 def relns (satisfying good homological/regularity hypotheses), minimality of dimension of the point (resp. line) scheme implies that it determines the def relns of the alg.

CONJECTURE

A = quadratic alg on n gens satisfying sufficient homological/regularity hypotheses. If ($n-3$)-linear scheme has minimal dimension, then it determines the def relns of A (whatever that means!).

On the other hand, even if dim($\mathcal{L}$) > 1 and $|\Gamma| = \infty$, this geometric data might still determine the def relns of the alg (examples are AS-reg algs of gldim 4 in [VV] & [SV] where point scheme $\cong$ quadric in $\mathbb{P}^3$ and does not determine def relns, but $\mathcal{L}$ determines def relns).

POINTS ON LINES

In the ATV geometry, a "point lies on a line" means that the corresponding point module is covered by the corresponding line module; i.e.,

$$p \in \ell \iff M(\ell) \twoheadrightarrow M(p).$$

PROPOSITION [Shelton V]

If A quadratic, noetherian, Aus-reg alg of gldim 4 such that $H(t) = (1-t)^{-4}$, then

(a) a line module that covers 4 non-isomorphic point modules covers infinitely many non-isomorphic point modules;

(b) every point module is covered by a line module;

(c) a point module that is covered by at most finitely many non-isomorphic line modules is covered by exactly 6 non-isomorphic line modules (counted with mult.). ■

QUESTIONS

1. A quadratic, 4 gens, 6 relns, $|\Gamma| < \infty$.

- What are the possible relative positions of the points of Γ?
- If A also AS-reg, etc, such that $\Gamma =$ graph of auto, what are the possible orbits? Would A be a finite module over its centre?

2. Does there exist A quadratic, with 4 gens, 6 relns, AS-reg such that

- $\Gamma = 20$ distinct points,
- dim(line scheme) $= 1$?

(If relax "distinct", answer = yes due to [VVW] alg above with 1 point of mult 20.)

3. Does there exist A quadratic, with 4 gens, 6 relns, AS-reg such that a point module is covered by precisely 6 non-isomorphic line modules?