GENERALIZING THE NOTION OF RANK TO NONCOMMUTATIVE QUADRATIC FORMS

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Abstract. In 2010, Cassidy and Vancliff extended the notion of a quadratic form on n generators to the noncommutative setting. In this article, we suggest a notion of rank for such noncommutative quadratic forms, where $n = 2$ or 3. Since writing an arbitrary quadratic form as a sum of squares fails in this context, our methods entail rewriting an arbitrary quadratic form as a sum of products. In so doing, we find analogs for 2×2 minors and determinant of a 3×3 matrix in this noncommutative setting.

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Introduction

Quadratic forms arise in many scientific fields, and consequently they been studied for many decades. Traditionally, the setting of quadratic forms has been commutative algebra and algebraic geometry, but, in recent years, quadratic forms have played a role in noncommutative algebra via their involvement in the study of graded Clifford algebras ([AL, L, VdB]). In this noncommutative setting, certain (commutative) quadratic forms associated to the graded Clifford algebra (GCA) determine a quadric system $\mathfrak{Q}$, and the regularity of the GCA and the degree of its generators and relations are completely determined by properties of $\mathfrak{Q}$. Moreover, elements of rank at most two within $\mathfrak{Q}$ determine properties of the point modules over the GCA ([VVW]).

In the last few years, an algebra that is a quantized analog of a GCA was introduced by Cassidy and Vancliff in [CV] and is called a graded skew Clifford algebra (GSCA). In this new setting, *noncommutative* quadratic forms (defined in [CV]) play a role relative to GSCAs that is identical to that played by (commutative) quadratic forms relative to GCAs. In particular, the regularity of the GSCA and the degree of its generators and relations are completely determined by properties of a certain *noncommutative* quadric system associated to the GSCA. Given $n \in \mathbb{N}$, GSCAs enable the relatively-easy production of quadratic regular algebras of global dimension n. Moreover, in [CV], many examples of GSCAs are given that are candidates for generic regular algebras of global dimension four, and, in [NVZ], it is shown that almost all quadratic regular algebras of global dimension three can be classified using GSCAs.

Given these recent developments, it is reasonable to attempt to extend the results in [VVW] for GCAs to GSCAs, but, in so doing, a notion of *rank* for noncommutative quadratic forms is needed. The purpose of this article is to suggest such a notion of rank on the noncommutative quadratic forms of [CV].

In Section 1, we establish notation to be used throughout the article and outline some technical issues that motivate our approach. Section 2 is devoted to the notion of rank for noncommutative quadratic forms on two generators. Our main result of that section is Proposition 2.4, which relates the factoring of a quadratic form Q on two generators as a perfect square to a noncommutative analog of the determinant of a 2×2 matrix associated to Q. That result motivates our definition of rank, in Definition 2.6, of a quadratic form on two generators. Since our noncommutative setting depends on the entries in a certain scalar matrix μ, our generalization of rank and determinant are called *μ-rank* and *μ-determinant*, respectively.

The case of quadratic forms on three generators is discussed in Section 3, with our main results relating the writing of an arbitrary quadratic form Q on three generators as a sum of products to analogs of the 2×2 minors, and determinant, of a 3×3 matrix associated to Q. In this section, our main result is Theorem 3.3, and our definition of μ-rank of a quadratic form on three generators is given in Definition 3.4.

We believe it should be possible to define μ-rank of a quadratic form on n generators, where $n \geq 4$, similar to our notion of μ-rank in Definition 3.4, where $n = 3$, but we expect the methods will be highly computational if the μ-rank is at least three. For n generators, where $n \geq 4$, and μ-rank at most two, the μ-rank can be defined in terms of factoring as in Definition 3.8. Fortunately, the results in [VVW] that promise to extend to the setting of GSCAs only entail quadratic forms of rank at most two; hence, the extension of those results is explored in [VV].

1. Noncommutative Quadratic Forms

In this section, we set up the noncommutative setting for our quadratic forms as defined in [CV, §1.2]. Our methods that are employed throughout the article to extend the traditional notion of rank to this noncommutative setting are discussed in §1.2.

1.1. **Definitions.**

Throughout the article, $\Bbbk$ denotes an algebraically closed field such that $\mathrm{char}(\Bbbk) \neq 2$, and $M(n, \Bbbk)$ denotes the vector space of $n \times n$ matrices with entries in $\Bbbk$. For a graded $\Bbbk$-algebra B, the span of the homogeneous elements in B of degree i will be denoted B_i, and the notation $T(V)$ will denote the tensor algebra on the vector space V. If C is any ring or vector space, then $C^\times$ will denote the nonzero elements in C. We use R to denote the polynomial ring on degree-one generators $x_1, \ldots, x_n$.

For $\{i,\ j\} \subset \{1, \ldots, n\}$, let $\mu_{ij} \in \Bbbk^\times$ satisfy the property that $\mu_{ij}\mu_{ji} = 1$ for all $i \neq j$. We write $\mu = (\mu_{ij}) \in M(n,\ \Bbbk)$. As in [CV], we write S for the quadratic $\Bbbk$-algebra on generators $z_1, \ldots, z_n$ with defining relations $z_j z_i = \mu_{ij} z_i z_j$ for all $i,\ j = 1, \ldots, n$, where $\mu_{ii} = 1$ for all i.

Definition 1.1. [CV, §1.2]

(a) With μ and S as above, a quadratic form Q is any element of S_2.
(b) A matrix $M \in M(n,\ \Bbbk)$ is called μ-symmetric if $M_{ij} = \mu_{ij} M_{ji}$ for all $i,\ j = 1, \ldots, n$.

We write $M^\mu(n,\ \Bbbk)$ for the set of μ-symmetric matrices in $M(n,\ \Bbbk)$. Clearly, if $\mu_{ij} = 1$ for all $i,\ j$, then $M^\mu(n,\ \Bbbk)$ consists of all symmetric matrices.

Henceforth, we assume that $\mu_{ii} = 1$ for all i.

As was shown in [CV, §1.2], the one-to-one correspondence between commutative quadratic forms and symmetric matrices has a counterpart in our setting, with the one-to-one correspondence being between noncommutative quadratic forms and μ-symmetric matrices. This correspondence is given as follows. For $M \in M^\mu(n,\ \Bbbk)$, let $\tilde{Q} = z^T M z \in T(S_1)_2$, where $z = (z_1, \ldots, z_n)^T$, and write Q for the image in S of $\tilde{Q}$; the element Q is the quadratic form corresponding to M. Conversely, if $Q = \sum_{i \leq j} \alpha_{ij} z_i z_j \in S$, where $\alpha_{ij} \in \Bbbk$ for all $i,\ j$, is a quadratic form, then the matrix (M_{ij}), where $M_{kk} = \alpha_{kk}$, $M_{ij} = 2^{-1}\alpha_{ij}$ and $M_{ji} = 2^{-1}\mu_{ji}\alpha_{ij}$ for all $i,\ j,\ k$ where $i < j$, is the μ-symmetric matrix corresponding to Q.

Remark 1.2. As was shown in [ATV1], if the point modules of S are parametrized by $\mathbb{P}^{n-1}$, then S is a twist (in the sense of [ATV2, §8]) of the polynomial ring by a graded degree-zero automorphism $\tau \in \text{Aut}(R)$ (see Definition 2.1 below). This case occurs if and only if $\mu_{ik} = \mu_{ij}\mu_{jk}$ for all $i,\ j,\ k$. This is the situation throughout Section 2, since there the assumption that $n = 2$ causes the point modules of S to be parametrized by $\mathbb{P}^1$.

1.2. **Technical Issues.**

Recall that, since $\Bbbk$ is algebraically closed, every (commutative) quadratic form $q \in R^\times$ can be written as $\sum_{i=1}^m y_i^2$, where $y_1, \ldots, y_m \in R_1$ are linearly independent and $m \in \mathbb{N}$ is unique; in this setting, the rank of q is defined to be m. However, if $Q \in S_2$ is a noncommutative quadratic form, then a direct generalization using a sum of squares leads to problems that are demonstrated by the following example.

Example 1.3. Suppose $n = 2$, $\mu_{12} = -1$ and $Q = z_1^2 + 2bz_1z_2 + cz_2^2$, where $b,\ c \in \Bbbk$. If $b \neq 0$, then $Q \neq \sum_{i=1}^m X_i^2$ for any $m \in \mathbb{N}$, where $X_i \in S_1$ for all i. Moreover, if $b = 0$, then $Q = z_1^2 + cz_2^2 = (z_1 + \alpha z_2)^2$, where $\alpha \in \Bbbk$, $\alpha^2 = c$. Hence, if $b \neq 0$, then a sum of squares is not possible; whereas if $b = 0$, then a sum of square terms is possible but the number of such terms is not unique.

Instead, for $n \leq 3$, we will model our notion of rank on the following facts concerning the rank of a (commutative) quadratic form $q \in R_2$:

(a) $\operatorname{rank}(q) = 0$ if and only if $q = 0$;
(b) $\operatorname{rank}(q) = 1$ if and only if $q = X^2$ for some $X \in R_1^\times$;
(c) $\operatorname{rank}(q) = 2$ if and only if $q = XY$ for some linearly independent $X,\ Y \in R_1$;
(d) $\operatorname{rank}(q) = 3$ if and only if $q = XY + Z^2$ for some linearly independent $X,\ Y,\ Z \in R_1$.

However, in our noncommutative setting, it is possible that $Q \in S_2$ might factor as both a perfect square and also as a product of linearly independent elements. This issue is highlighted in the next example.

Example 1.4. If $Q = z_1^2 + 6z_1z_2 + 4z_2^2 \in S_2$, with $\mu_{12} = 2$, then $Q = (z_1 + 2z_2)^2 = (z_1 + z_2)(z_1 + 4z_2)$.

2. Rank of Quadratic Forms on Two Generators

In this section, we consider noncommutative quadratic forms on two generators as defined in Section 1.1, and introduce a notion of rank, called μ-rank, in Definition 2.6, on such quadratic forms that extends the notion of rank of a commutative quadratic form. We also introduce in Definition 2.3 an analog of the determinant of a 2×2 matrix.

Throughout this section we suppose $n = 2$. As mentioned in Remark 1.2, we use the notion of *twist* in this section, which is defined as follows.

Definition 2.1. [ATV2, §8] Let $B = \bigoplus_{m \geq 0} B_m$ be a quadratic algebra and let ϕ be a graded degree-zero automorphism of B. The twist B^ϕ of B by ϕ is the vector space $\bigoplus_{m \geq 0} B_m$ with a new multiplication $*$ defined as follows: if $x, y \in B_1$, then $x * y = x\phi(y)$, where the right-hand side is computed using the original multiplication in B.

By Remark 1.2, in this section, the algebra S is a twist of the polynomial ring R by a graded, degree-zero automorphism $\tau \in \operatorname{Aut}(R)$. In this section, we denote multiplication in S by $*$ and the action of τ by $r^\tau = \tau(r)$ for all $r \in R$. By [N, Lemma 5.6], we may choose τ to be given by

$$\tau(z_1) = \mu_{12} z_1 \quad \text{and} \quad \tau(z_2) = z_2. \tag{$*$}$$

Lemma 2.2. *Suppose $n = 2$. If $Q \in S_2$ is a quadratic form, then Q factors in at most two distinct ways.*

Proof. Suppose $Q = r_1 * r_2 = r_3 * r_4 = r_5 * r_6$ in S, where $r_i \in S_1$ for all i. Using τ given above in $(*)$, it follows that $Q = r_1 r_2^\tau = r_3 r_4^\tau = r_5 r_6^\tau$ in R. However, in R, the element Q factors in at most two distinct ways, so, without loss of generality, we may assume $r_5 \in \Bbbk^\times r_3$ and $r_6 \in \Bbbk^\times r_4$. Hence, in S, Q factors in at most two ways. ■

For the rest of this section, we will be concerned with a quadratic form $az_1 * z_1 + 2bz_1 * z_2 + cz_2 * z_2 \in S_2$, where $a,\ b,\ c \in \Bbbk$. As explained in §1.1, to such a quadratic form is associated a μ-symmetric matrix $M = \begin{bmatrix} a & b \\ \mu_{21} b & c \end{bmatrix}$. It will be useful to use an analog of the determinant function on M in the next result.

Definition 2.3. Let $D : M^\mu(2,\ \Bbbk) \to \Bbbk$ be given by

$$D(M) = 4b^2 - (1 + \mu_{12})^2 ac, \quad \text{where} \quad M = \begin{bmatrix} a & b \\ \mu_{21} b & c \end{bmatrix};$$

we call $D(M)$ the μ-determinant of M.

We remark that if $S = R$, that is, if $\mu_{12} = 1$, then $D(M) = -4\det(M)$.

Proposition 2.4. *Let $Q = az_1 * z_1 + 2bz_1 * z_2 + cz_2 * z_2 \in S_2^\times$, where $a,\ b,\ c \in \Bbbk$, be a quadratic form with associated μ-symmetric matrix $M \in M^\mu(2,\ \Bbbk)$.*

(a) *There exists $L_1, L_2 \in S_1$ such that $Q = L_1 * L_2$ in S.*
(b) *There exists $L \in S_1$ such that $Q = L * L$ in S if and only if $D(M) = 0$.*
(c) *The element Q factors uniquely, up to a nonzero scalar multiple, in S if and only if $b^2 = \mu_{12} ac$.*

Proof. Viewing $Q \in R$, we have $Q = a\mu_{12} z_1^2 + 2bz_1 z_2 + cz_2^2$.

(a) Since Q factors in R, we have $Q = r_1 r_2$, where $r_i \in R_1 = S_1$ for all i. Thus, in S, $Q = r_1 * \tau^{-1}(r_2)$, which proves (a).

(b) If $Q = r * r$ in S, for some $r \in S_1$, then

$$Q = rr^\tau = \mu_{12}\alpha_1^2 z_1^2 + (1 + \mu_{12})\alpha_1\alpha_2 z_1 z_2 + \alpha_2^2 z_2^2$$

in R, where $r = \alpha_1 z_1 + \alpha_2 z_2$ for some $\alpha_1,\ \alpha_2 \in \Bbbk$. Comparing coefficients, it follows that this situation occurs if and only if $2b = (1 + \mu_{12})\alpha_1\alpha_2$, where $\alpha_1^2 = a$ and $\alpha_2^2 = c$. Hence, $Q = r * r$ for some $r \in S_1$ implies that $D(M) = 0$. Conversely, if $D(M) = 0$, then $2b = (1 + \mu_{12})\beta$, where $\beta \in \Bbbk$ and $\beta^2 = ac$. If also $ac = 0$, then (b) follows; whereas if $ac \neq 0$, then we may choose $\alpha_1,\ \alpha_2 \in \Bbbk$ such that $\alpha_1^2 = a$ and $\alpha_2 = \beta/\alpha_1$, which implies that $Q = r * r$ in S, where $r = \alpha_1 z_1 + \alpha_2 z_2$.

(c) A quadratic form factors uniquely in S if and only if it factors uniquely in R, and the latter occurs if and only if the discriminant is zero. Since the discriminant of $a\mu_{12}z_1^2 + 2bz_1z_2 + cz_2^2 \in R_2$ belongs to $\Bbbk^\times(b^2 - \mu_{12}ac)$, the result follows. ∎

Corollary 2.5. *Let Q be as in Proposition 2.4.*

(a) *Suppose Q does not factor uniquely. If $ac = 0$, then $Q \in \langle z_i \rangle$ for some $i \in \{1,\ 2\}$; whereas if $ac \neq 0$, then*

$$Q = \left(z_1 + \frac{cz_2}{b + H} \right) * (a\, z_1 + [b + H]\, z_2\,),$$

where $H^2 = b^2 - \mu_{12}ac$.

(b) *Suppose Q factors uniquely, up to a nonzero scalar multiple, in S. If $b = 0$, then $Q \in \Bbbk^\times z_i^2$ for some $i \in \{1,\ 2\}$; whereas if $b \neq 0$, then*

$$Q = b^{-1}(bz_1 + cz_2) * (az_1 + bz_2).$$

Proof.

(a) If $ac = 0$, the result in (a) clearly holds. If $ac \neq 0$, we may write $Q = a^{-1}(az_1 + \alpha z_2) * (az_1 + \beta z_2)$, where $\alpha,\ \beta \in \Bbbk^\times$. Comparing coefficients, we find $ac = \alpha\beta$ and $2b = \beta + \mu_{12}\alpha$. Solving for β yields $\beta = b + H$, where $H^2 = b^2 - \mu_{12}ac$. Since $\alpha = ac/(b + H)$, part (a) follows.

(b) By Proposition 2.4(c), $b^2 = \mu_{12}ac$. Thus, if $b = 0$, the result in (b) clearly holds. If $b \neq 0$, then $ac \neq 0$, so part (a) applies with $H = 0$. ∎

Proposition 2.4 suggests the following generalization of the rank of a quadratic form on two generators.

Definition 2.6. Let $Q = az_1 * z_1 + 2bz_1 * z_2 + cz_2 * z_2 \in S_2$, where a, b, $c \in \Bbbk$, let $M \in M^{\mu}(2,\ \Bbbk)$ be the μ-symmetric matrix associated to Q and let $D : M^{\mu}(2,\ \Bbbk) \to \Bbbk$ be defined as in Definition 2.3. If $n = 2$, we define μ-rank : $S_2 \to \mathbb{N}$ as follows:

(a) if $Q = 0$, we define $\mu\text{-rank}(Q) = 0$;

(b) if $Q \neq 0$ and $D(M) = 0$, we define $\mu\text{-rank}(Q) = 1$;

(c) if $D(M) \neq 0$, we define $\mu\text{-rank}(Q) = 2$.

Example 2.7. If Q is the quadratic form in Example 1.4, then $\mu\text{-rank}(Q) = 1$.

Corollary 2.8. *Let $n = 2$. If $Q \in S_2^{\times}$, then μ-rank$(Q) = 1$ if and only if $Q = L * L$ for some $L \in S_1^{\times}$.*

Proof. Combine Definition 2.6 and Proposition 2.4(b). ■

3. Rank of Quadratic Forms on Three Generators

In this section, we explore further the notion of rank on noncommutative quadratic forms, and extend the results of the previous section concerning μ-rank of quadratic forms on two generators to quadratic forms on three generators. Our main result of this section is Theorem 3.3, which uses analogs of the determinant and minors of a 3×3 matrix to describe factoring properties of a quadratic form. Our definition of μ-rank of a noncommutative quadratic form on three generators is given in Definition 3.4.

Since $n = 3$ throughout this section, the methods of Section 2 cannot be employed directly since the algebra S, where $n \geq 3$, need not be a twist of a polynomial ring. In particular, we henceforth use juxtaposition to denote the multiplication in S.

Proposition 3.1. *If $Q = az_1^2 + bz_2^2 + cz_3^2 + 2dz_1z_2 + 2ez_1z_3 + 2fz_2z_3 \in S_2$, where $a, \ldots, f \in \Bbbk$, is a quadratic form, then $Q = L_1L_2 + L_3^2$ for some $L_1,\ L_2,\ L_3 \in S_1$.*

Proof. If $a = b = c = e = 0$, then the result clearly holds. Moreover, if $a = b = c = 0 \neq e$, then

$$Q = 2(z_1 + \alpha z_2)(dz_2 + ez_3) - 2\alpha d z_2^2,$$

where $\alpha \in \Bbbk$ and $\alpha e = f$. Hence, by symmetry, it suffices to prove the result in the case $a \neq 0$. Thus, for simplicity, we henceforth assume that $a = 1$.

If $\mu_{12} \neq -1 \neq \mu_{13}$, then

$$Q = Q' + \left(z_1 + \frac{2d}{1+\mu_{12}} z_2 + \frac{2e}{1+\mu_{13}} z_3 \right)^2,$$

where $Q' \in \Bbbk z_2^2 + \Bbbk z_3^2 + \Bbbk z_2 z_3$. Applying Proposition 2.4(a) to Q' implies the result in this case.

Suppose $\mu_{12} = -1 \neq \mu_{13}$. If $c \neq 0$ or $e \neq 0$, then there exists $\delta \in \Bbbk$ such that $\delta^2 = c$ and $2e \neq (1 + \mu_{13})\delta$. In this case,

$$Q = (z_1 + \gamma z_2 + \delta z_3)^2 + (z_1 + \alpha z_2)(2dz_2 + \beta z_3),$$

where $\alpha, \ldots, \delta \in \Bbbk$ satisfy

$$\delta^2 = c, \quad \beta = 2e - (1 + \mu_{13})\delta \neq 0, \quad \gamma^2 = b - 2d\alpha \quad \text{and} \quad (1 + \mu_{23})\gamma\delta + \alpha\beta = 2f.$$

However, if $c = 0 = e$, then $Q = (z_1 + \epsilon z_2)^2 - 2z_2(dz_1 - fz_3)$, where $\epsilon \in \Bbbk$, $\epsilon^2 = b$. Similarly, if $\mu_{12} \neq -1 = \mu_{13}$.

It remains to consider $\mu_{12} = -1 = \mu_{13}$. If $e \neq 0$, then there exist solutions α, β, $\gamma \in \Bbbk$ to the equations

$$\alpha^2 + 2d\gamma = b, \quad \beta^2 = c \quad \text{and} \quad (1 + \mu_{23})\alpha\beta + 2e\gamma = 2f,$$

so that

$$Q = (z_1 + \alpha z_2 + \beta z_3)^2 + 2(z_1 + \gamma z_2)(dz_2 + ez_3).$$

On the other hand, if $e = 0$, then $Q = (z_1 + \delta z_3)^2 + (2dz_1 + bz_2 + 2\mu_{32} f z_3)z_2$, where $\delta \in \Bbbk$, $\delta^2 = c$. ■

In order to generalize Proposition 2.4 and Definition 2.6 to the three-generator case, we introduce analogs of the determinant and 2×2 minors of a 3×3 matrix.

Definition 3.2. Let $M = \begin{bmatrix} a & d & e \\ \mu_{21}d & b & f \\ \mu_{31}e & \mu_{32}f & c \end{bmatrix} \in M^\mu(3, \Bbbk)$ and, for $1 \leq i \leq 8$, define the functions $D_i : M^\mu(3, \Bbbk) \to \Bbbk$ by

$$D_1(M) = 4d^2 - (1+\mu_{12})^2 ab, \qquad D_4(M) = 2(1+\mu_{23})de - (1+\mu_{12})(1+\mu_{13})af,$$
$$D_2(M) = 4e^2 - (1+\mu_{13})^2 ac, \qquad D_5(M) = 2(1+\mu_{12})ef - (1+\mu_{13})(1+\mu_{23})cd,$$
$$D_3(M) = 4f^2 - (1+\mu_{23})^2 bc, \qquad D_6(M) = 2(1+\mu_{13})df - (1+\mu_{12})(1+\mu_{23})be,$$
$$D_7(M) = (\mu_{23}cd^2 - 2def + be^2)(\mu_{13}\mu_{21}cd^2 - 2def + \mu_{12}\mu_{23}\mu_{31}be^2),$$
$$D_8(M) = \mu_{21}(d+X)(e-Y) + \mu_{23}\mu_{31}(d-X)(e+Y) - 2af,$$

where $X^2 = d^2 - \mu_{12}ab$ and $Y^2 = e^2 - \mu_{13}ac$. We call $D_1, \ldots, D_6$ the 2×2 μ-minors of M. The functions D_7 and D_8 will play a role analogous to that of the determinant of M and so could be called the μ-determinants of M, even though D_8 is not a polynomial in the entries of M. (Attempting to convert D_8 to a polynomial leads to unwieldy polynomials such as the one given after Theorem 3.3.)

Theorem 3.3. *Let $Q = az_1^2 + bz_2^2 + cz_3^2 + 2dz_1z_2 + 2ez_1z_3 + 2fz_2z_3 \in S_2$, where $a, \ldots, f \in \Bbbk$, and let $M \in M^\mu(3, \Bbbk)$ be the μ-symmetric matrix associated to Q.*

(a) *There exists $L \in S_1$ such that $Q = L^2$ if and only if $D_i(M) = 0$ for all $i = 1, \ldots, 6$.*

(b) (i) *If $a = 0$, then there exists $L_1, L_2 \in S_1$ such that $Q = L_1L_2$ if and only if $D_7(M) = 0$;*

(ii) *if $a \neq 0$, then there exists $L_1, L_2 \in S_1$ such that $Q = L_1L_2$ if and only if $D_8(M) = 0$ for some X and Y satisfying $X^2 = d^2 - \mu_{12}ab$ and $Y^2 = e^2 - \mu_{13}ac$.*

Proof. By Proposition 3.1, $Q = L_1L_2 + L_3^2$ for some $L_1, L_2, L_3 \in S_1$.

(a) Suppose there exist $\alpha_1, \alpha_2, \alpha_3 \in \Bbbk$ such that $Q = (\alpha_1z_1 + \alpha_2z_2 + \alpha_3z_3)^2$. Comparing coefficients, it follows that

$$\text{(i)}\ \ 2d = (1+\mu_{12})\alpha_1\alpha_2, \qquad \text{(iv)}\ \ a = \alpha_1^2,$$
$$\text{(ii)}\ \ 2e = (1+\mu_{13})\alpha_1\alpha_3, \qquad \text{(v)}\ \ b = \alpha_2^2,$$
$$\text{(iii)}\ \ 2f = (1+\mu_{23})\alpha_2\alpha_3, \qquad \text{(vi)}\ \ c = \alpha_3^2,$$

so $D_i(M) = 0$ for $i = 1, 2, 3$. Moreover, from equations (i)-(iv), we have

$$\begin{aligned} 4de(1+\mu_{23}) &= (2d)(2e)(1+\mu_{23}) \\ &= (1+\mu_{12})(1+\mu_{13})(1+\mu_{23})\alpha_1^2\alpha_2\alpha_3 \\ &= (1+\mu_{12})(1+\mu_{13})2af, \end{aligned}$$

so $D_4(M) = 0$. By symmetry, $D_i(M) = 0$ for $i = 5, 6$.

Conversely, suppose that $D_i(M) = 0$ for all $i = 1, \ldots, 6$. If $a = 0$, then $d = 0 = e$, since $D_1(M) = 0 = D_2(M)$. In this case, $Q \in \Bbbk z_2^2 + \Bbbk z_3^2 + \Bbbk z_2 z_3$, so Proposition 2.4(b) applies to Q (since $D_3(M) = 0$), and so $Q = L^2$, where $L \in S_1$. Thus, to complete the proof of (a), we may assume $a \neq 0$.

Since $D_i(M) = 0$ for $i = 1, 2, 3$, there exist $w_1, w_2, w_3 \in \Bbbk$ such that

$$2d = (1+\mu_{12})w_1, \qquad 2e = (1+\mu_{13})w_2, \qquad 2f = (1+\mu_{23})w_3, \tag{vii}$$

where $w_1^2 = ab$, $w_2^2 = ac$, $w_3^2 = bc$. Since $a \neq 0$, let $Q' = a^{-1}(az_1 + w_1 z_2 + w_2 z_3)^2 \in S_2$. By (vii), it follows that

$$Q' = az_1^2 + bz_2^2 + cz_3^2 + 2dz_1z_2 + 2ez_1z_3 + a^{-1}(1+\mu_{23})w_1w_2z_2z_3.$$

If $(1+\mu_{23})bc = 0$, then $Q' = Q$ and (a) follows. If $\mu_{12} = -1$, then w_1 may be chosen so that $Q' = Q$; similarly for w_2 if $\mu_{13} = -1$. Hence, we may assume

$$(1+\mu_{12})(1+\mu_{13})(1+\mu_{23})bc \neq 0. \tag{viii}$$

Moreover,

$$\begin{aligned}(1+\mu_{12})(1+\mu_{13})(1+\mu_{23})w_1w_2 &= 4de(1+\mu_{23}), && \text{using (vii)}\\ &= 2(1+\mu_{12})(1+\mu_{13})af, && \text{as } D_4(M)=0\\ &= (1+\mu_{12})(1+\mu_{13})(1+\mu_{23})aw_3, && \text{using (vii).}\end{aligned}$$

Thus, since (viii) holds, $w_1w_2 = aw_3$, from which it follows that $Q' = Q$, which completes the proof of (a).

(b)(i) Suppose $a = 0$. If also $d = 0$, then, by Proposition 2.4(a), Q factors if and only if $be = 0$, and the latter holds if and only if $D_7(M) = 0$. Since a similar argument applies if instead $a = 0 = e$, we may assume $de \neq 0$. Let $Q_1, \ Q_2 \in S_2$ be given by

$$\begin{aligned}Q_1 &= 2[z_1 + (2d)^{-1}bz_2 + (2e)^{-1}cz_3][dz_2 + ez_3]\\ &= bz_2^2 + cz_3^2 + 2dz_1z_2 + 2ez_1z_3 + (bed^{-1} + cd\mu_{23}e^{-1})z_2z_3,\end{aligned}$$

$$\begin{aligned}Q_2 &= 2[d\mu_{21}z_2 + e\mu_{31}z_3][z_1 + b\mu_{12}(2d)^{-1}z_2 + c\mu_{13}(2e)^{-1}z_3]\\ &= bz_2^2 + cz_3^2 + 2dz_1z_2 + 2ez_1z_3 + [be\mu_{12}\mu_{23}(d\mu_{13})^{-1} + cd\mu_{13}(e\mu_{12})^{-1}]z_2z_3.\end{aligned}$$

If Q factors, then the coefficients of z_2^2, z_3^2, z_1z_2 and z_1z_3 of Q imply that $Q = Q_1$ or $Q = Q_2$. By comparing the coefficients of z_2z_3 in each case, we find $D_7(M) = 0$. Conversely, if $D_7(M) = 0$, then $Q = Q_1$ or $Q = Q_2$, so Q factors.

(b)(ii) Suppose $a \neq 0$ and that Q factors. We may write

$$Q = a^{-1}(az_1 + \alpha_2 z_2 + \alpha_3 z_3)(az_1 + \beta_2 z_2 + \beta_3 z_3),$$

for some α_2, α_3, β_2, $\beta_3 \in \Bbbk$. Comparing coefficients, we have

$$ab = \alpha_2\beta_2, \qquad 2d = \beta_2 + \mu_{12}\alpha_2, \qquad 2e = \beta_3 + \mu_{13}\alpha_3, \tag{ix}$$

$$ac = \alpha_3\beta_3, \qquad 2af = \alpha_2\beta_3 + \mu_{23}\alpha_3\beta_2. \tag{x}$$

Equations (ix) imply that $ab = \alpha_2(2d - \mu_{12}\alpha_2)$, and so $\alpha_2 = \mu_{21}(d+X)$, where $X^2 = d^2 - \mu_{12}ab$. Similarly, $\alpha_3 = \mu_{31}(e+Y)$, where $Y^2 = e^2 - \mu_{13}ac$.

From the second equation in (x), it follows that

$$\begin{aligned} 2af &= \alpha_2(2e - \mu_{13}\alpha_3) + \mu_{23}\alpha_3(2d - \mu_{12}\alpha_2) \\ &= \mu_{21}(d+X)(e-Y) + \mu_{23}\mu_{31}(d-X)(e+Y), \end{aligned}$$

where X and Y are as above. Hence, $D_8(M) = 0$ for some X and Y such that $X^2 = d^2 - \mu_{12}ab$ and $Y^2 = e^2 - \mu_{13}ac$.

Conversely, suppose $a \neq 0$ and that $D_8(M) = 0$ for some X and Y satisfying $X^2 = d^2 - \mu_{12}ab$ and $Y^2 = e^2 - \mu_{13}ac$. Let $Q' \in S_2$, where

$$\begin{aligned} Q' &= a^{-1}[az_1 + \mu_{21}(d+X)z_2 + \mu_{31}(e+Y)z_3][az_1 + (d-X)z_2 + (e-Y)z_3] \\ &= az_1^2 + bz_2^2 + cz_3^2 + 2dz_1z_2 + 2ez_1z_3 + \\ &\qquad + a^{-1}[\mu_{21}(d+X)(e-Y) + \mu_{23}\mu_{31}(e+Y)(d-X)]z_2z_3. \end{aligned}$$

The last coefficient equals $2f$, since $D_8(M) = 0$, and so $Q' = Q$, which completes the proof of (b)(ii). ■

We remark that, in Theorem 3.3(b)(ii), converting the equation $D_8(M) = 0$ to a polynomial equation yields, at best, a user-unfriendly polynomial equation of degree six:

$$\begin{aligned} 0 = {} & (\mu_{13} + \mu_{12}\mu_{23})^4 a^2b^2c^2 + 64\mu_{12}\mu_{13}\mu_{23}d^2e^2f^2 + \\ & +16(\mu_{12}^2\mu_{13}^2a^2f^4 + \mu_{12}^2\mu_{23}^2b^2e^4 + \mu_{13}^2\mu_{23}^2c^2d^4) + \\ & +16(\mu_{13}^2 + \mu_{12}^2\mu_{23}^2)(\mu_{12}abe^2f^2 + \mu_{13}acd^2f^2 + \mu_{23}bcd^2e^2) + \\ & -32(\mu_{13} + \mu_{12}\mu_{23})(\mu_{12}\mu_{13}adef^3 + \mu_{12}\mu_{23}bde^3f + \mu_{13}\mu_{23}cd^3ef) + \\ & -8(\mu_{13} + \mu_{12}\mu_{23})^2(\mu_{12}\mu_{13}a^2bcf^2 + \mu_{12}\mu_{23}ab^2ce^2 + \mu_{13}\mu_{23}abc^2d^2) + \\ & -8(\mu_{13}^3 - 5\mu_{12}\mu_{13}^2\mu_{23} - 5\mu_{12}^2\mu_{13}\mu_{23}^2 + \mu_{12}^3\mu_{23}^3)abcdef. \end{aligned}$$

Theorem 3.3 suggests the following generalization of μ-rank in Definition 2.6 to the three-generator case.

Definition 3.4. Let $Q = az_1^2 + bz_2^2 + cz_3^2 + 2dz_1z_2 + 2ez_1z_3 + 2fz_2z_3 \in S_2$, where $a, \dots, f \in \Bbbk$, with $a = 0$ or 1, let $M \in M^\mu(3, \Bbbk)$ be the μ-symmetric matrix associated to Q and let $D_i : M^\mu(3, \Bbbk) \to \Bbbk$, for $i = 1, \dots, 8$, be defined as in Definition 3.2. If $n = 3$, we define the function μ-rank : $S_2 \to \mathbb{N}$ as follows:

(a) if $Q = 0$, we define μ-rank$(Q) = 0$;

(b) if $Q \neq 0$ and if $D_i(M) = 0$ for all $i = 1, \dots, 6$, we define μ-rank$(Q) = 1$;

(c) if $D_i(M) \neq 0$ for some $i = 1, \dots, 6$ and if

$$(1 - a)D_7(M) + aD_8(M) = 0,$$

we define μ-rank$(Q) = 2$;

(d) if $(1 - a)D_7(M) + aD_8(M) \neq 0$, we define μ-rank$(Q) = 3$.

Example 3.5. If $Q = (2z_1+z_2+8z_3)^2 = (2\mu_{12}z_1+z_2+8z_3)(2\mu_{21}z_1+z_2+8z_3)$, where $\mu_{12} = \mu_{13}$, then μ-rank$(Q) = 1$, by Definition 3.4 and Theorem 3.3(a).

Corollary 3.6. *Let $n = 3$.*

(a) *If $Q \in S_2^\times$, then μ-rank$(Q) \leq 2$ if and only if $Q = L_1L_2$ for some $L_1, L_2 \in S_1^\times$.*

(b) *If $Q \in S_2^\times$, then μ-rank$(Q) = 1$ if and only if $Q = L^2$ for some $L \in S_1^\times$.*

Proof. The result follows from Theorem 3.3. ■

The following result gives simplified versions of D_7 and D_8 in the special case where S is a twist of the polynomial ring (see Remark 1.2).

Corollary 3.7. *Let $n = 3$. If S is a twist of the polynomial ring by an automorphism (see Remark 1.2), then*

$$D_7(M) = (\mu_{23}cd^2 - 2def + be^2)^2 \qquad \textit{and} \qquad D_8(M) = 2[\,\mu_{21}(de - XY) - af\,],$$

where $X^2 = d^2 - \mu_{12}ab$ and $Y^2 = e^2 - \mu_{13}ac$.

Proof. By Remark 1.2, $\mu_{13} = \mu_{12}\mu_{23}$, so the result follows. ■

The results in this article suggest that generalizing the notion of rank to quadratic forms on four or more generators is likely to be very computation heavy. However, in the spirit of Corollary 3.6, one could define μ-rank one, respectively μ-rank two, by simply using factoring as follows.

Definition 3.8. Let $n \in \mathbb{N}$, $n > 0$, and let $Q \in S_2^\times$.

(a) If $Q = L^2$ for some $L \in S_1^\times$, we define μ-rank$(Q) = 1$.

(b) If $Q \neq L^2$ for any $L \in S_1^\times$, but $Q = L_1 L_2$ where L_1, $L_2 \in S_1^\times$, we define μ-rank$(Q) = 2$.

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