

# SOME QUANTUM $\mathbb{P}^3$ S WITH ONE POINT

BRAD SHELTON AND MICHAELA VANCLIFF \*

Department of Mathematics  
University of Oregon  
Eugene, Oregon 97403-1222  
shelton@math.uoregon.edu    vancliff@math.uoregon.edu

ABSTRACT. We study a certain 1-parameter family of non-commutative graded regular algebras of global dimension four which were introduced by Vancliff, Van Rompay and Willaert in [12]. Most members of the family have a singleton point scheme of multiplicity twenty. Our objective is to analyse the point scheme of these algebras and the coordinate ring of the point scheme. In particular, we prove that the point scheme determines the defining relations and that its coordinate ring is a Frobenius algebra.

## INTRODUCTION

The concept of a regular, graded algebra was defined in [2] and the connected, regular, graded algebras of global dimension three were classified in [2, 3, 4, 9, 10]. The regular algebras of global dimension four have not yet been classified; indeed it is still uncertain what characteristics are exhibited by a generic, regular algebra of global dimension four.

The classification of the regular algebras of global dimension three employed geometric techniques first introduced in [3]. Certain graded modules (point modules, line modules, etc.) were associated to geometric objects (points, lines, etc.) in some projective space. This inspired a definition of  $\text{Proj}A$  in [1] for any graded algebra  $A$  generated by degree one elements, and the term “quantum  $\mathbb{P}^2$ ” was adopted in [1] for  $\text{Proj}A$  where  $A$  is a graded regular algebra of global dimension three generated by degree one elements. Understanding the different types of quantum  $\mathbb{P}^2$  that exist led to the classification in [3, 4] of the regular algebras of global dimension three which are generated by degree one elements.

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Henceforth, unless otherwise stated, all algebras herein are assumed to be connected,  $\mathbb{N}$ -graded algebras generated by degree one elements.

The above geometric methods and ideas suggest that in order to classify the regular algebras of global dimension four, one perhaps should first define a notion of “quantum  $\mathbb{P}^3$ ” and classify all such entities. However, it is uncertain what should be the definition of a quantum  $\mathbb{P}^3$ . Ideally it should be a generalisation of a quantum  $\mathbb{P}^2$ ; namely,  $\text{Proj}A$  where  $A$  is a regular algebra of global dimension four, but should  $A$  satisfy additional hypotheses? It is conceivable that regular algebras of global dimension three have certain characteristics important to the concept of a quantum  $\mathbb{P}^2$  which are not automatically shared by regular algebras of global dimension four. To date, no definition of a quantum  $\mathbb{P}^3$  has been given, although numerous examples of regular algebras of global dimension four have been studied.

In [3, §3] it is shown that a certain scheme parametrises the point modules. In [11] this scheme is called the “point scheme,” and it gives easily accessible points of  $\text{Proj}$  of the algebra. The scheme structure of the point scheme was a major tool in the classification of the subclass of algebras of global dimension four which were considered in [11]. Moreover, it is shown in [11, Theorem 1.10] that, under certain hypotheses, a quadratic, regular algebra of global dimension four has point scheme which is the graph of an automorphism of some scheme. As such, a first potential approach to classifying such algebras is to classify those quadratic, regular algebras of global dimension four whose point scheme is the graph of an automorphism. To date, the different kinds of point scheme which arise in this situation are unknown.

On the other hand, if  $A$  is a quadratic algebra, then the zero locus  $\Gamma_A$  of the defining relations of  $A$  contains the point scheme of  $A$  (see [3]). M. Van den Bergh has shown that if  $A$  is a generic, quadratic algebra on four generators with six defining relations, then  $\Gamma_A$  consists of precisely twenty distinct points; he has also shown that the variety of line modules over such an algebra is 1-dimensional (see [12]). However, although examples of quadratic, regular algebras of global dimension four are known whose point scheme consists of twenty distinct points (see [12, 13]), no such examples are known whose variety of line modules is 1-dimensional. If such an example could be found, it would be a candidate for a generic, regular algebra of global dimension four.

One potential method to construct such an example is to understand similar examples with as “small” a point scheme as possible, and then try to use that understanding to build one with twenty points. In [12], a family of quadratic, algebras  $A(q, \lambda)$  on four generators with six defining relations was constructed where  $q, \lambda \in k^\times$ . If  $q^4 = 1$ , then  $A(q, \lambda)$  is regular of global dimension four for all  $\lambda$ . If  $q^4 = 1$ , but  $q \neq 1$ , then the point scheme of  $A(q, \lambda)$  consists of one point of multiplicity twenty, and if  $q^2 = -1$ , then the variety of line modules is 1-dimensional. Our objective in this paper is to analyse the point scheme of  $A(q, \lambda)$  and its coordinate ring.

In §1, the point scheme is explicitly computed in Proposition 1.9. In §2, we prove in Theorem 2.2 that the point scheme of  $A(q, \lambda)$  determines the defining relations of  $A(q, \lambda)$  if  $q^4 = 1$  and  $q \neq 1$ . This is somewhat surprising given that the point scheme has only one closed point, but the multiplicity of the point being twenty is encoded in the coordinate ring which has dimension twenty as a vector space. The analysis of a degree two form which vanishes on the point scheme is carried out in the 20-dimensional coordinate ring.

Since the point scheme of  $A(q, \lambda)$  consists of one point of multiplicity twenty, it seems reasonable to expect some symmetry to reside in its coordinate ring. In §2, we verify this by proving in Theorem 2.4 that the coordinate ring of a finite point scheme of a quadratic algebra on four generators with six defining relations is a Frobenius algebra; that is, the coordinate ring and its vector space dual are isomorphic as modules over the coordinate ring.

On the other hand, it was demonstrated in [11] that in general the point scheme encodes an insufficient amount of data for classifying the regular algebras of global dimension four, although it was sufficient for regular algebras of global dimension three. Moreover, most of the algebras  $A(q, \lambda)$  (even those which are not regular) have defining relations which are determined by their point scheme. This supports the fact that in general the point scheme does not distinguish between quadratic algebras on four generators. Since the point scheme parametrises the point modules, it is conceivable that, for regular algebras of global dimension four, the relevant geometric data is a scheme that parametrises the line modules (see [8]).

## 1. THE POINT SCHEME

In this section we consider a certain family of quadratic algebras  $A(q, \lambda)$  where  $q, \lambda \in k^\times$ . If  $q^3 \neq 1$ , then  $A(q, \lambda)$  has a singleton point scheme of multiplicity twenty, whereas if  $q^4 = 1$  the algebra  $A(q, \lambda)$  is regular.

### 1.1. Definitions.

Throughout the paper,  $k$  denotes an algebraically closed field of characteristic different from two. Unless otherwise stated, all algebras herein are assumed to be connected,  $\mathbb{N}$ -graded  $k$ -algebras generated by degree one elements. Let  $S(\mathbb{P}^3)$  denote the homogeneous coordinate ring of  $\mathbb{P}^3$ .

**Definition 1.1.** [12, §3] Let  $A(q, \lambda)$  denote the  $k$ -algebra on four generators  $x_1, x_2, x_3$  and  $x_4$  with defining relations

$$\begin{aligned} x_1x_2 - qx_2x_1 &= \lambda x_4^2, & x_2x_3 &= qx_3x_2, \\ x_1x_3 - qx_3x_1 &= x_2^2, & x_3x_4 &= qx_4x_3, \\ x_1x_4 - qx_4x_1 &= x_3^2, & x_4x_2 &= qx_2x_4, \end{aligned}$$

where  $q, \lambda \in k^\times$ .

The specialisation  $A(-1, 1)$  is a graded Clifford algebra ([12]). It is shown in [12] that if  $q^4 = 1$ , then  $A(q, \lambda)$  is an iterated Ore extension, so that it has Hilbert series the same as that of  $S(\mathbb{P}^3)$ , and it is a regular algebra of global dimension four in the following sense.

**Definition 1.2.** [2, Page 171] A connected  $\mathbb{N}$ -graded  $k$ -algebra  $B$  is called regular if

- (a) the global (homological) dimension of  $B$  ( $\text{gldim}(B)$ ) is finite,
- (b) the Gelfand-Kirillov dimension of  $B$  ( $\text{GKdim}(B)$ ) is finite, and
- (c)  $B$  is Gorenstein; that is,  $\text{Ext}_B^q(k, B) = \delta_n^q k$  where  $n = \text{gldim}(B)$ .

Furthermore, if  $q^4 = 1$ , then  $A(q, \lambda)$  is noetherian, Auslander-regular and satisfies the Cohen-Macaulay property (see [12]).

**Remarks 1.3.**

- (a) For every  $q, \lambda \in k^\times$ , we have  $A(q^{-1}, \lambda) \cong A(q, \lambda)^{\text{op}}$ .
- (b) If  $q^4 \neq 1$ , then  $A(q, \lambda)$  is not regular, which can be seen by computing the Hilbert series of its Koszul dual. This is because a quadratic, regular algebra of global dimension four on four generators is Koszul and has Hilbert series  $H(t) = (1 - t)^{-4}$  (see [7]).

Let  $A$  denote a quadratic algebra on four generators which has six defining relations, and let  $\Gamma_A$  denote the zero locus in  $\mathbb{P}^3 \times \mathbb{P}^3$  of its defining relations. M. Van den Bergh has shown that if  $A$  is generic, then  $\Gamma_A$  consists of exactly twenty points. It follows from his argument that if  $\Gamma_A$  is finite, then it consists of twenty points counted with multiplicity.

**Notation 1.4.** Let  $\Gamma(q, \lambda)$  denote the scheme-theoretic zero locus in  $\mathbb{P}^3 \times \mathbb{P}^3$  of the defining relations of  $A(q, \lambda)$ .

**Lemma 1.5.** *If  $q^3 \neq 1$ , then  $\Gamma(q, \lambda)$  consists of one point in  $\mathbb{P}^3 \times \mathbb{P}^3$  of multiplicity 20; in the coordinates determined by the  $x_i$ , the point is  $((1, 0, 0, 0), (1, 0, 0, 0))$ . If  $q^3 = 1$ , then  $\Gamma(q, \lambda)$  is infinite.*

**Proof.** The result follows from the proof of [12, Lemma 3.3], and the multiplicity follows from the preceding paragraph. ■

**Definition 1.6.** [3, §3] A point module over  $A(q, \lambda)$  is a cyclic, graded module  $M$  which has Hilbert series  $H_M(t) = (1 - t)^{-1}$ .

**Lemma 1.7.** *If  $q^4 = 1$ , then  $\Gamma(q, \lambda)$  is the graph of an automorphism  $\sigma_{q, \lambda}$  of a subscheme  $P(q, \lambda)$  of  $\mathbb{P}^3$ . In this case,  $\Gamma(q, \lambda)$  parametrises the point modules of  $A(q, \lambda)$ .*

**Proof.** If  $q^4 = 1$ , but  $q \neq 1$ , then  $A(q, \lambda)$  satisfies the relevant hypotheses of [5, Theorem 4.1.3], and, by Lemma 1.5,  $\Gamma(q, \lambda)$  satisfies the relevant hypotheses of the same theorem; whence, in this case, the result follows from [5, Theorem 4.1.3]. If  $q = 1$ , then [11, Theorem 1.10] applies to  $A(1, \lambda)$ , so the result holds. The last statement is a consequence of [3, §3]. ■

**Definition 1.8.** [11] In the case that  $\Gamma(q, \lambda)$  is the graph of an automorphism, we refer to  $P(q, \lambda)$ , or  $(P(q, \lambda), \sigma_{q, \lambda})$  or  $\Gamma(q, \lambda)$  as the point scheme of  $A(q, \lambda)$ . In keeping with [5], we use the term point variety to refer to the reduced scheme determined by the point scheme.

In the next subsection, the point scheme  $(P(q, \lambda), \sigma_{q, \lambda})$  is computed explicitly.

## 1.2. Computation of the Point Scheme.

To analyse the point scheme we use the methods of [3, §3] as follows.

Fix  $q, \lambda \in k^\times$ . Let  $\{f_i\}_{i=1}^6$  denote the defining relations of  $A(q, \lambda)$  and let  $M$  be a  $6 \times 4$  matrix which satisfies the equation

$$M(x_i)^T = (f_i)^T;$$

the matrix  $M$  depends on  $q$  and  $\lambda$ , and  $M$  is unique up to row operations. By definition, a point  $(a, b) \in \mathbb{P}^3 \times \mathbb{P}^3$  belongs to  $\Gamma(q, \lambda)$  if and only if  $M|_a b^T = 0$ ; that is, if and only if  $\text{rank}(M|_a) < 4$ . Let  $I(q, \lambda)$  denote the ideal in  $S(\mathbb{P}^3)$  generated by the fifteen  $4 \times 4$  minors of  $M$ . It follows from Lemma 1.7 that if  $q^4 = 1$ , then  $P(q, \lambda) \cong \mathcal{V}(I(q, \lambda))$ . On the other hand, by Lemma 1.5, if  $q^3 \neq 1$ , then  $\Gamma(q, \lambda)$  is an affine scheme. This means that if  $q^4 = 1$  but  $q \neq 1$ , then the stalk of the structure sheaf  $\mathcal{O}_{P(q, \lambda)}$  of  $P(q, \lambda)$  over the unique point  $(1, 0, 0, 0)$  is given by the degree zero component of

$$\frac{S(\mathbb{P}^3)[x_1^{-1}]}{I_1(q, \lambda)},$$

where  $I_1(q, \lambda)$  is the image of  $I(q, \lambda)$  in  $S(\mathbb{P}^3)[x_1^{-1}]$ . This stalk is a local ring of dimension twenty as a  $k$ -vector space, since the unique point of the point scheme has multiplicity twenty; since the point scheme is affine, the stalk is its coordinate ring.

Our interest is in the structure of this finite dimensional local ring and its automorphism  $\sigma_{q, \lambda}$ . As such, we assume that

$$q^4 = 1 \quad \text{and} \quad q \neq 1$$

throughout the rest of the paper,

Under this assumption, the automorphism  $\sigma_{q, \lambda}$  is an automorphism of the coordinate ring of the point scheme. To compute the action of  $\sigma_{q, \lambda}$ , observe that since  $\Gamma(q, \lambda) \cong P(q, \lambda)$  (by Lemma 1.7), the stalk  $\mathcal{O}_{P(q, \lambda), (1, 0, 0, 0)}$  is isomorphic to the stalk of  $\mathcal{O}_{\Gamma(q, \lambda)}$  over the point  $((1, 0, 0, 0), (1, 0, 0, 0))$ . To identify the latter, set  $Z'_i = x_i \otimes 1$  and  $Y'_i = 1 \otimes x_i$  for all  $i = 1, \dots, 4$  in the defining relations of  $A(q, \lambda)$ , and then set  $Z_i = Z'_i/Z'_1$  and  $Y_i = Y'_i/Y'_1$  for all  $i = 2, 3, 4$ . This yields that the stalk of  $\mathcal{O}_{\Gamma(q, \lambda)}$  over  $((1, 0, 0, 0), (1, 0, 0, 0))$  is isomorphic to the commutative algebra

$$\frac{k[Z_2, Z_3, Z_4, Y_2, Y_3, Y_4]}{J(q, \lambda)},$$

where  $J(q, \lambda)$  is the ideal generated by the six elements

$$\begin{aligned} Y_2 - qZ_2 - \lambda Y_4 Z_4, & & Z_2 Y_3 - qZ_3 Y_2, \\ Y_3 - qZ_3 - Y_2 Z_2, & & Z_3 Y_4 - qZ_4 Y_3, \\ Y_4 - qZ_4 - Y_3 Z_3, & & Z_4 Y_2 - qZ_2 Y_4. \end{aligned}$$

By definition of  $\Gamma(q, \lambda)$  and  $\sigma_{q, \lambda}$ , it follows that  $\sigma_{q, \lambda}$  is given by  $\sigma_{q, \lambda}(Y_i) = Z_i$  for all  $i$ . Hence, computing  $\sigma_{q, \lambda}$  entails rewriting the  $Z_i$  in terms of the  $Y_i$ .

**Proposition 1.9.** *If  $q^4 = 1$  but  $q \neq 1$ , then  $A(q, \lambda)$  has a singleton point scheme  $\Gamma(q, \lambda)$  of multiplicity 20 which is the graph of an automorphism  $\sigma_{q, \lambda}$  of the subscheme  $P(q, \lambda) = \mathcal{V}(I(q, \lambda))$  where  $\sigma_{q, \lambda} \in \text{Aut}(\mathcal{O}_{P(q, \lambda), (1, 0, 0, 0)})$  is given by*

$$\begin{aligned} \sigma_{q, \lambda}(y_2) &= q^3 y_2 - \lambda q^2 y_4^2 + \lambda q y_3^2 y_4, \\ \sigma_{q, \lambda}(y_3) &= q^3 y_3 - q^2 y_2^2 + \lambda q y_2 y_4^2, \\ \sigma_{q, \lambda}(y_4) &= q^3 y_4 - q^2 y_3^2 + q y_2^2 y_3, \end{aligned}$$

where  $y_i = x_i/x_1 \in \mathcal{O}_{P(q, \lambda), (1, 0, 0, 0)}$  for  $i = 2, 3, 4$ .

**Remark 1.10.** By Remarks 1.3(a), if  $q^2 = -1$ , then the formulae for  $\sigma_{q^3, \lambda}$  and  $(\sigma_{q, \lambda})^{-1}$  are the same on the generators  $y_i$  of the respective stalks.

**Proof.** By Lemma 1.7,  $\Gamma(q, \lambda) \cong P(q, \lambda)$ , so that

$$\frac{k[Z_2, Z_3, Z_4, Y_2, Y_3, Y_4]}{J(q, \lambda)} \cong \left( \frac{S(\mathbb{P}^3)[x_1^{-1}]}{I_1(q, \lambda)} \right)_0,$$

where  $Y_i \mapsto x_i/x_1 = y_i$  for all  $i = 2, 3, 4$ . Solving for  $Z_2$  we find that

$$\begin{aligned} Z_2 &= q^{-1}(Y_2 - \lambda Y_4 Z_4) \\ &= q^{-1}Y_2 - \lambda q^{-2}Y_4(Y_4 - Y_3 Z_3) \\ &= q^{-1}Y_2 - \lambda q^{-2}Y_4^2 + \lambda q^{-3}Y_3 Y_4(Y_3 - Y_2 Z_2) \\ &= q^{-1}Y_2 - \lambda q^{-2}Y_4^2 + \lambda q^{-3}Y_3^2 Y_4 - \lambda q^{-3}Y_2 Y_3 Y_4 Z_2. \end{aligned}$$

However,

$$Y_2 Y_3 Y_4 Z_2 = q^{-1} Y_2^2 Y_3 Z_4 = q^{-2} Y_2^2 Y_4 Z_3 = q^{-3} Y_2 Y_3 Y_4 Z_2,$$

and since, by assumption,  $q^3 \neq 1$ , it follows that

$$Z_2 = q^3 Y_2 - \lambda q^2 Y_4^2 + \lambda q Y_3^2 Y_4,$$

(since  $q^4 = 1$ ); similarly for  $Z_3$  and  $Z_4$ . This gives a formula for  $Z_i$  for each  $i$ , and since  $Z_i = \sigma_{q, \lambda}(Y_i)$  and  $Y_i \mapsto y_i$  for all  $i$ , the result follows.  $\blacksquare$

**Remark 1.11.** By Remarks 1.3(a),  $\sigma_{-1, \lambda} = (\sigma_{-1, \lambda})^{-1}$ , so that the order of  $\sigma_{-1, \lambda}$  is two for all  $\lambda \in k^\times$ . It may be verified by computation that if  $q^2 = -1$ , then the order of  $\sigma_{q, \lambda}$  is four for all  $\lambda \in k^\times$ .

Let  $R$  denote a quadratic, regular algebra of global dimension four which has a finite point scheme. The examples known of such algebras  $R$  ([12, 13]) are finite modules over their centres, and the automorphism of the point scheme has finite order. It is not known whether or not in

general  $R$  need be a finite module over its centre, nor if the automorphism of the point scheme need have finite order.

## 2. THE DEFINING RELATIONS

The notation used in this section is defined in §1 unless otherwise specified, and the parameter  $q$  in Definition 1.1 is assumed to satisfy  $q^4 = 1$  and  $q \neq 1$ . It is proved in Theorem 2.2 that, for every  $\lambda \in k^\times$ , the defining relations of  $A(q, \lambda)$  are determined by its point scheme. Moreover, we prove in Theorem 2.4 that the coordinate ring of the point scheme is a Frobenius algebra.

Let  $f \in A(q, \lambda)_1 \otimes_k A(q, \lambda)_1$  and suppose that  $f$  vanishes on  $\Gamma(q, \lambda)$ . Our intention is to show that  $f$  belongs to the span of the defining relations of  $A(q, \lambda)$ . To do this we consider the image of  $f$  in the stalk of  $\mathcal{O}_{P(q, \lambda)}$  over  $(1, 0, 0, 0)$ ; its image in this stalk is zero due to the assumption that  $f$  vanishes on  $\Gamma(q, \lambda)$ . Hence, we first compute a basis for this stalk.

Let  $A$  denote any quadratic, regular algebra of global dimension four on four generators  $z_1, z_2, z_3, z_4$ , with six defining relations  $g_1, \dots, g_6$ . Let  $\mathcal{M}$  be a  $6 \times 4$  matrix determined by the equation  $(g_i)^T = \mathcal{M}(z_i)^T$ . Suppose that the zero locus  $\Gamma$  of the defining relations of  $A$  in  $\mathbb{P}^3 \times \mathbb{P}^3$  is finite. For  $i = 1, 2$ , let  $\pi_i : \mathbb{P}^3 \times \mathbb{P}^3 \rightarrow \mathbb{P}^3$  denote the projection map onto the  $i$ th coordinate. By [7], these assumptions imply that the Hilbert series of  $A$  is  $H_A(t) = (1 - t)^{-4}$ , and, by [3, Proposition 3.6], the finiteness of  $\Gamma$  implies that the reduced scheme determined by  $\pi_1(\Gamma)$  is isomorphic to the reduced scheme determined by  $\pi_2(\Gamma)$ . Hence by [5, Theorem 4.1.3],  $\Gamma$  is the graph of an automorphism of a subscheme  $\mathcal{P}$  of  $\mathbb{P}^3$ . It follows that  $\mathcal{P}$  is an affine scheme, so that we may assume (by rechoosing the  $z_i$  if necessary) that  $\mathcal{P}$  is contained in the affine open set  $\mathbb{P}^3 \setminus \mathcal{V}(z_1)$ ; thus,  $\mathcal{P} = \text{Spec}(\mathcal{O}_{\mathcal{P}}(\mathcal{P}))$ . Moreover,

$$\mathcal{O}_{\mathcal{P}}(\mathcal{P}) = \left( \frac{k[z_1, \dots, z_4][z_1^{-1}]}{\langle \text{images of } 4 \times 4 \text{ minors of } \mathcal{M} \rangle} \right)_0.$$

In this situation, we have the following result.

**Lemma 2.1.** *Let  $\mathcal{M}$  and  $\mathcal{P}$  be defined as in the preceding paragraph. If the fifteen  $4 \times 4$  minors of  $\mathcal{M}$  are linearly independent, then  $\mathcal{O}_{\mathcal{P}}(\mathcal{P})$  has a vector space basis consisting of all monomials in  $k[\bar{z}_2, \bar{z}_3, \bar{z}_4]_{\leq 3}$ , where  $\bar{z}_i = z_i/z_1$  for all  $i = 2, 3, 4$ .*

**Proof.** Let  $p_1, \dots, p_{15}$  denote the  $4 \times 4$  minors of  $\mathcal{M}$ . We may write each minor in the form  $p_i = p_{i2} - z_1 p_{i1}$ , where  $p_{i2} \in k[z_2, z_3, z_4]$ . Suppose that  $\{p_{i2}\}_{i=1}^{15}$  is a linearly independent set, so that it is a basis for the 15-dimensional space  $k[z_2, z_3, z_4]_4$ . Let  $g$  be a degree four monomial in  $k[z_2, z_3, z_4]$ ; that is,  $g \in \sum k p_{i2}$ . However, the image of each  $p_{i2}$  in  $\mathcal{O}_{\mathcal{P}}(\mathcal{P})$  is either zero or is a polynomial of degree three or less in the variables  $\bar{z}_2, \bar{z}_3, \bar{z}_4$ , where  $\bar{z}_i = z_i/z_1$  for  $i = 2, 3, 4$ .

It follows that the image of  $g$  in  $\mathcal{O}_{\mathcal{P}}(\mathcal{P})$  is a linear combination of polynomials of degree three or less in the variables  $\bar{z}_2, \bar{z}_3, \bar{z}_4$ . Since

$$\dim_k(k[\bar{z}_2, \bar{z}_3, \bar{z}_4]_{\leq 3}) = 20 = \dim_k(\mathcal{O}_{\mathcal{P}}(\mathcal{P})),$$

the result follows.

It remains to prove that  $\{p_{i2}\}_{i=1}^{15}$  is a linearly independent set. If this were false, then there would exist scalars  $\alpha_1, \dots, \alpha_{15}$  not all zero such that  $\sum \alpha_i p_{i2} = 0$ ; whence,  $\sum \alpha_i p_i$  (which is nonzero by hypothesis) would be divisible by  $z_1$ . In other words, there would exist  $h \in k[z_1, \dots, z_4]_3$  such that  $h \neq 0$  and  $z_1 h$  belongs to the ideal  $I$  generated by the fifteen  $4 \times 4$  minors of  $\mathcal{M}$ ; in particular,  $\deg(h) = 3$  and  $h \notin I$ . We may write  $I = \bigcap_{j=1}^m Q_j$ , for some  $m \in \mathbb{N}$ , where the  $Q_j$  are homogeneous primary ideals. Since  $h \notin I$ , we would have  $h \notin Q_n$  for some  $n$ , but  $z_1 h \in Q_n$ , so  $z_1 \in \sqrt{Q_n}$  (since  $Q_n$  is primary). It would then follow that  $z_1$  vanishes on a point of  $\mathcal{P}$ , which would contradict our choice of  $z_1, \dots, z_4$ .  $\blacksquare$

In order to apply Lemma 2.1 in the proof of Theorem 2.2, one should check that the fifteen  $4 \times 4$  minors defining  $I(q, \lambda)$ , namely

$$\begin{aligned} & -qx_1x_2^3 - q(1-q)x_1^2x_2x_3 + \lambda q^2x_3^3x_4 - \lambda qx_2^2x_4^2 + \lambda q^2x_1x_3x_4^2, \\ & -qx_2^2x_3^2 - qx_1x_3^3 + q^2x_1x_2^2x_4 - q(1-q)x_1^2x_3x_4 + \lambda q^2x_2x_4^3, \\ & q^2x_2^3x_3 + q^2x_1x_2x_3^2 - q(1-q)x_1^2x_2x_4 - \lambda qx_3^2x_4^2 - \lambda qx_1x_4^3, \\ & \quad qx_2^3x_3 + q(1-q)x_1x_2x_3^2 - \lambda q^3x_3^2x_4^2, \\ & \quad qx_3^3x_4 - q^3x_2^2x_4^2 + q(1-q)x_1x_3x_4^2, \\ & -q^3x_2^2x_3^2 + q(1-q)x_1x_2^2x_4 + \lambda qx_2x_4^3, \\ & -q^2x_2^4 - q^2(1-q)x_1x_2^2x_3 + \lambda qx_2x_3x_4^2, \\ & \quad q^2x_3^4 - qx_2^2x_3x_4 + q^2(1-q)x_1x_3^2x_4, \\ & \quad qx_2x_3^2x_4 - q^2(1-q)x_1x_2x_4^2 - \lambda q^2x_4^4, \\ & q^3x_2^3x_4 - q(1-q^2)x_1x_2x_3x_4 - \lambda q^2x_3x_4^3, \\ & -q^3x_2x_3^3 + q^2x_2^3x_4 + q(1-q^2)x_1x_2x_3x_4, \\ & q^2x_2x_3^3 + q(1-q^2)x_1x_2x_3x_4 - \lambda q^3x_3x_4^3, \\ & \quad -q(1-q^3)x_2^2x_3x_4, \\ & \quad -q(1-q^3)x_2x_3^2x_4, \\ & \quad -q(1-q^3)x_2x_3x_4^2, \end{aligned}$$

are linearly independent. This fact is easily verified by computation.

**Theorem 2.2.** *If  $q^4 = 1$  but  $q \neq 1$ , then*

$$A(q, \lambda) \cong \frac{k\langle x_1, \dots, x_4 \rangle}{\langle f \in (\oplus kx_i) \otimes_k (\oplus kx_i) : f|_{\Gamma(q, \lambda)} = 0 \rangle}.$$

**Proof.** Let  $f \in (\oplus kx_i) \otimes_k (\oplus kx_i)$  and suppose that  $f|_{\Gamma(q, \lambda)} = 0$ . We will prove that  $f$  belongs to the span of the defining relations of  $A(q, \lambda)$ .

We may write  $f = \sum \alpha_{ij} x_i \otimes x_j$ , where the  $\alpha_{ij} \in k$ . By comparing  $f$  with the defining relations of  $A(q, \lambda)$ , we may assume that  $\alpha_{ij} = 0$  for all  $i > j$ , so that we need to show that  $f = 0$  to prove the result. Firstly, since  $f$  vanishes on the point  $((1, 0, 0, 0), (1, 0, 0, 0))$ , we have that  $\alpha_{11} = 0$ .

Moreover,  $f$  vanishes on the point scheme  $\Gamma(q, \lambda)$ , so the image  $\bar{f}$  of  $f$  in the stalk of  $\mathcal{O}_{\Gamma(q, \lambda)}$  over the point  $((1, 0, 0, 0), (1, 0, 0, 0))$  is zero. That is,

$$0 = \bar{f} = \sum_{i=2}^4 \alpha_{1i} Y_i + \sum_{2 \leq i < j} \alpha_{ij} Z_i Y_j,$$

where the  $Z_i$  and  $Y_j$  are as in §1.2. Moreover, the image  $\tilde{f}$  of  $f$  in the stalk of  $\mathcal{O}_{P(q, \lambda)}$  over the point  $(1, 0, 0, 0)$  is also zero. In particular, we have that

$$0 = \tilde{f} = \sum_{i=2}^4 \alpha_{1i} y_i + \sum_{2 \leq i < j} \alpha_{ij} \sigma_{q, \lambda}(y_i) y_j,$$

by Proposition 1.9 and its proof. However,  $\Gamma(q, \lambda)$  is finite, so, since the reduced scheme of  $P(q, \lambda)$  has only one point, by Lemma 2.1, the stalk of  $\mathcal{O}_{P(q, \lambda)}$  over the point  $(1, 0, 0, 0)$  has a basis consisting of the monomials in  $y_2, y_3, y_4$  of degree three or less. Rewriting  $\tilde{f}$  in terms of this basis (via a computer program, such as *Mathematica*, or otherwise), one obtains

$$\begin{aligned} \tilde{f} = & \alpha_{12} y_2 + \alpha_{13} y_3 + \alpha_{14} y_4 + q^3(\alpha_{22} y_2^2 + \alpha_{33} y_3^2 + \alpha_{44} y_4^2) + \\ & + (1 + q^3)(\alpha_{23} y_2 y_3 + \alpha_{34} y_3 y_4) - (1 + q)\alpha_{24} y_2 y_4 + \\ & - q^2(\alpha_{33} y_2^2 y_3 + \alpha_{44} y_3^2 y_4 + \alpha_{22} \lambda y_2 y_4^2) + \\ & + \beta_q(\alpha_{23} y_2^3 - \alpha_{24} y_2 y_3^2 + \alpha_{34} y_3^3) + \\ & + \beta_{q^3}(\alpha_{34} y_2^2 y_4 + \alpha_{23} \lambda y_3 y_4^2 - q^2 \alpha_{24} \lambda y_4^3), \end{aligned}$$

where

$$\beta_r = \begin{cases} \frac{1}{2} & \text{if } r = -1 \\ \frac{1+r}{2} & \text{if } r^2 = -1 \end{cases}.$$

Since  $\tilde{f} = 0$ , we may set the coefficients of the basis elements equal to zero in the expression for  $\tilde{f}$ . This yields that the remaining unknown  $\alpha_{ij}$  are all equal to zero. Hence  $f = 0$  which completes the proof.  $\blacksquare$

Theorem 2.2 is perhaps somewhat surprising given that the point variety is so small, being only one point. This result highlights the importance of the point scheme over the point variety,

since the multiplicity of the point being twenty plays a key role in the proof. It is conceivable that a similar result holds for all quadratic, connected, regular  $k$ -algebras of global dimension four which have a finite point scheme.

Given the symmetry of the point scheme, one might expect that its coordinate ring would exhibit some symmetry too. Indeed, the following results prove that the coordinate ring is a Frobenius algebra.

Let  $R$  denote any finite dimensional commutative algebra. By standard theory,  $R^*$  is an  $R$ -module via the action defined by  $(r \cdot \eta)(r') = \eta(r'r)$  for all  $r, r' \in R$ , for all  $\eta \in R^*$ . In this context,  $R$  is called a Frobenius algebra if  $R^*$  and  $R$  are isomorphic as  $R$ -modules.

We thank M. Van den Bergh for drawing our attention to the following argument. The proof of the next result is included due to lack of a suitable reference.

**Proposition 2.3.** *Let  $S$  denote the polynomial ring on  $n$  variables and let  $I$  denote an ideal of  $S$  generated by  $n$  elements. If  $\dim_k(S/I) < \infty$ , then  $S/I$  is a Frobenius algebra.*

**Proof.** Suppose that  $S/I$  is finite dimensional. By the Chinese Remainder theorem,  $S/I$  is a finite direct sum of local rings of the form  $S_{\bar{\mathfrak{m}}}/I_{\bar{\mathfrak{m}}}$  where  $\bar{\mathfrak{m}}$  denotes the preimage in  $S$  of a maximal ideal  $\mathfrak{m}$  of  $S/I$  and  $I_{\bar{\mathfrak{m}}}$  denotes the image of  $I$  in the localisation  $S_{\bar{\mathfrak{m}}}$ . Since  $\text{Kdim}(S_{\bar{\mathfrak{m}}}/I_{\bar{\mathfrak{m}}}) = 0$  for every  $\bar{\mathfrak{m}}$ , we have that  $S_{\bar{\mathfrak{m}}}/I_{\bar{\mathfrak{m}}}$  is a complete intersection (by [6, Theorem 31(iii)]). It follows that  $S_{\bar{\mathfrak{m}}}/I_{\bar{\mathfrak{m}}}$  is an injective module over itself for all  $\bar{\mathfrak{m}}$ . By dualizing, one obtains that  $(S_{\bar{\mathfrak{m}}}/I_{\bar{\mathfrak{m}}})^*$  is a finite dimensional projective module for all  $\bar{\mathfrak{m}}$  and thus is free; since it is indecomposable, it is isomorphic to  $S_{\bar{\mathfrak{m}}}/I_{\bar{\mathfrak{m}}}$ . Hence,  $S_{\bar{\mathfrak{m}}}/I_{\bar{\mathfrak{m}}}$  is Frobenius for every  $\bar{\mathfrak{m}}$ , and the result follows. ■

**Theorem 2.4.** *Let  $A$  denote any quadratic  $k$ -algebra on four generators with six defining relations. If the zero locus  $\Gamma$  in  $\mathbb{P}^3 \times \mathbb{P}^3$  of the defining relations of  $A$  is finite, then the coordinate ring of  $\Gamma$  is a Frobenius algebra.*

**Proof.** If  $\Gamma$  is finite, then it is an affine scheme and its coordinate ring is finite dimensional. In this case, coordinates may be chosen so that the coordinate ring of  $\Gamma$  is a quotient of the polynomial ring on six variables by six relations. The result follows from Proposition 2.3. ■

In particular, Theorem 2.4 applies to  $A(q, \lambda)$  for all  $q, \lambda \in k^\times$  such that the zero locus of the defining relations is finite (for example, if  $q$  is a fourth root of unity but not equal to one).

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