POINT MODULES OVER REGULAR GRADED SKEW CLIFFORD ALGEBRAS

Michaela Vancliff*
Department of Mathematics, P.O. Box 19408
University of Texas at Arlington, Arlington, TX 76019-0408
vancliff@uta.edu
www.uta.edu/math/vancliff

and

Padmini P. Veerapen†
Department of Mathematics
Tennessee Technological University, Cookeville, TN 38505
pveerapen@tntech.edu

Abstract. Results of Vancliff, Van Rompay and Willaert in 1998 ([VVW]) prove that point modules over a regular graded Clifford algebra (GCA) are determined by (commutative) quadrics of rank at most two that belong to the quadric system associated to the GCA. In 2010, in [CV], Cassidy and Vancliff generalized the notion of a GCA to that of a graded skew Clifford algebra (GSCA). The results in this article show that the results of [VVW] may be extended, with suitable modification, to GSCAs. In particular, using the notion of μ-rank introduced recently by the authors in [VV], the point modules over a regular GSCA are determined by (noncommutative) quadrics of μ-rank at most two that belong to the noncommutative quadric system associated to the GSCA.

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Introduction

The notion of a graded skew Clifford algebra (GSCA) was introduced in [CV], and it is an algebra that may be viewed as a quantized analog of a graded Clifford algebra (GCA). In [CV], it was shown that many of the results that hold for GCAs have analogous counterparts in the context of GSCAs. In particular, homological and algebraic properties of a GSCA are determined by properties of a certain quadric system associated to the GSCA. The importance of GSCAs was highlighted in [NVZ], where they were shown to play a critical role in the classification of the quadratic Artin-Schelter regular (AS-regular) algebras of global dimension three, and, again, in [CV], where some families of GSCAs were presented that contain candidates for generic AS-regular algebras of global dimension four (that is, they have twenty distinct point modules and a one-dimensional line scheme); indeed, the only algebras known to date that are candidates for generic AS-regular algebras of global dimension four are GSCAs. Hence, it is thought that GSCAs might play a critical role in the classification of the quadratic AS-regular algebras of global dimension four and greater. The reader is referred to [AS, ATV1, ATV2] for results concerning AS-regular algebras and their associated geometric data, and to [SV, VVW] for results concerning GCAs and their associated geometric data.

Consequently, it is reasonable to attempt to extend the results in [VVW] concerning point modules over GCAs to point modules over GSCAs. Hence, our main objective in this article is to generalize [VVW, Theorem 1.7]. That result states, in part, that if the number, N, of point modules over a regular GCA is finite, then $N = 2r_2 + r_1$, where r_j is the number of elements of rank j that belong to the projectivization of a certain quadric system associated to the GCA (see Theorem 2.10 for the precise statement). We achieve our objective in Theorem 2.12, where the notion of μ-rank (introduced in [VV]) is used in place of the traditional notion of rank. However, [VVW, Theorem 1.7] also states, in part, that if $N < \infty$, then $r_1 \in \{0, 1\}$. We present examples in the last section that demonstrate that this part of [VVW, Theorem 1.7] appears not to have an obvious counterpart in the setting of GSCAs.

Although the flow of this article follows that of [VVW, §1], many of our results require methods of proof that differ substantially from those used in [VVW, §1], since the proofs in [VVW] make use of standard results concerning symmetric matrices and the general linear group. This article consists of two sections: in Section 1, notation and terminology are defined, while Section 2 is devoted to proving our main result, which is given in Theorem 2.12.

1. Graded Skew Clifford Algebras

In this section, we define the notion of a graded skew Clifford algebra from [CV], and give the relevant results from [CV] needed in Section 2.

Throughout the article, $\Bbbk$ denotes an algebraically closed field such that $\mathrm{char}(\Bbbk) \neq 2$, and $M(n, \Bbbk)$ denotes the vector space of $n \times n$ matrices with entries in $\Bbbk$. For a graded $\Bbbk$-algebra B, the span of the homogeneous elements in B of degree i will be denoted B_i, and $T(V)$ will denote the tensor algebra on the vector space V. If C is a vector space, then $C^\times$ will denote the nonzero elements in C, and C^* will denote the vector space dual of C.

For $\{i, j\} \subset \{1, \ldots, n\}$, let $\mu_{ij} \in \Bbbk^\times$ satisfy the property that $\mu_{ij}\mu_{ji} = 1$ for all i, j where $i \neq j$. We write $\mu = (\mu_{ij}) \in M(n, \Bbbk)$. As in [CV], we write S for the quadratic $\Bbbk$-algebra on generators $z_1, \ldots, z_n$ with defining relations $z_j z_i = \mu_{ij} z_i z_j$ for all $i, j = 1, \ldots, n$, where $\mu_{ii} = 1$ for all i. We set $U \subset S_1 \otimes_\Bbbk S_1$ to be the span of the defining relations of S and write $z = (z_1, \ldots, z_n)^T$.

Definition 1.1. [CV, §1.2]

(a) With μ and S as above, a (noncommutative) quadratic form is defined to be any element of S_2. By identifying $\mathbb{P}(S_1^*)$ with $\mathbb{P}^{n-1}$, the subscheme of the zero locus $Z \subset \mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$ of U consisting of all points in Z on which a quadratic form q vanishes is called the quadric determined by q.

(b) A matrix $M \in M(n, \Bbbk)$ is called μ-symmetric if $M_{ij} = \mu_{ij} M_{ji}$ for all i, $j = 1, \ldots, n$.

Henceforth, we assume $\mu_{ii} = 1$ for all i, and write $M^\mu(n, \Bbbk)$ for the vector space of μ-symmetric $n \times n$ matrices with entries in $\Bbbk$. By [CV], we have $M^\mu(n, \Bbbk) \cong S_2$, as vector spaces, via $M \mapsto z^T M z \in S$. This map mirrors the isomorphism between symmetric matrices and commutative quadratic forms.

Notation 1.2. Let $\tau : \mathbb{P}(M^\mu(n, \Bbbk)) \to \mathbb{P}(S_2)$ be defined by $\tau(M) = z^T M z$. Henceforth, we fix $M_1, \ldots, M_n \in M^\mu(n, \Bbbk)$. For each $k = 1, \ldots, n$, we fix representatives $q_k = \tau(M_k)$.

Definition 1.3. [CV] A *graded skew Clifford algebra* $A = A(\mu, M_1, \ldots, M_n)$ associated to μ and $M_1, \ldots, M_n$ is a graded $\Bbbk$-algebra on degree-one generators $x_1, \ldots, x_n$ and on degree-two generators $y_1, \ldots, y_n$ with defining relations given by:

(a) degree-two relations: $x_i x_j + \mu_{ij} x_j x_i = \displaystyle\sum_{k=1}^{n} (M_k)_{ij} y_k$ for all $i, j = 1, \ldots, n$, and

(b) degree-three and degree-four relations that guarantee the existence of a normalizing sequence $\{y_1', \ldots, y_n'\}$ that spans $\Bbbk y_1 + \cdots + \Bbbk y_n$.

Remark 1.4. If A is a graded skew Clifford algebra (GSCA), then [CV, Lemma 1.13] implies that $y_i \in (A_1)^2$ for all $i = 1, \ldots, n$ if and only if $M_1, \ldots, M_n$ are linearly independent. Thus, hereafter, we assume that $M_1, \ldots, M_n$ are linearly independent.

By [CV], the degree of the defining relations of A and certain homological properties of A are intimately tied to certain geometric data associated to A as follows.

Definition 1.5.

(a) [CV] The span of quadratic forms $Q_1, \ldots, Q_m \in S_2$ will be called the *quadric system* associated to $Q_1, \ldots, Q_m$. If a quadric system is given by a normalizing sequence in S, then it is called a *normalizing quadric system.*

(b) [CVc] We define a *left base point* of a quadric system $\mathfrak{Q} \subset S_2$ to be any left base-point module over $S/\langle\mathfrak{Q}\rangle$; that is, to be any 1-critical graded left module over $S/\langle\mathfrak{Q}\rangle$ that is generated by its homogeneous degree-zero elements and which has Hilbert series $H(t) = c/(1-t)$, for some $c \in \mathbb{N}$. We say a quadric system is *left base-point free* if it has no left base points. Similarly, for right base point, etc.

If S is commutative, then its only base-point modules are point modules, so, in this setting, the above definitions of "base point" and "base-point free" coincide with their commutative counterparts.

By [CVc, Corollary 11], a normalizing quadric system $\mathfrak{Q}$ is left base-point free if and only if $\dim_{\Bbbk}(S/\langle\mathfrak{Q}\rangle) < \infty$. Hence, such a quadric system is left base-point free if and only if it is right base-point free. In particular, the adjectives "right" and "left" may be dropped when referring to a *normalizing* quadric system being base-point free.

Theorem 1.6. [CV, CVc] *For all $k = 1, \ldots, n$, let M_k and q_k be as in Notation 1.2. A graded skew Clifford algebra $A = A(\mu, M_1, \ldots, M_n)$ is a quadratic, Auslander-regular algebra of global dimension n that satisfies the Cohen-Macaulay property with Hilbert series $1/(1-t)^n$ if and only if the quadric system associated to $\{q_1, \ldots, q_n\}$ is normalizing and base-point free; in this case, A is a noetherian Artin-Schelter regular domain and is uniquely determined (up to isomorphism) by the data $\{\mu, M_1, \ldots, M_n\}$.*

Remark 1.7.

(a) Henceforth, we assume that the quadric system associated to $\{q_1, \ldots, q_n\}$ is normalizing and base-point free. By Theorem 1.6, this assumption allows us to write $A = T(V)/\langle W\rangle$, where $V = S_1^*$ and $W \subseteq T(V)_2$, and write the Koszul dual of A as $T(S_1)/\langle W^\perp\rangle = S/\langle q_1, \ldots, q_n\rangle$. In this setting, $\{x_1, \ldots, x_n\}$ is the dual basis in V to the basis $\{z_1, \ldots, z_n\}$ of S_1, and we write $\sum_{i,j} \alpha_{ijm}(x_i x_j + \mu_{ij} x_j x_i) = 0$ for the defining relations of A, where $\alpha_{ijm} \in \Bbbk$

for all i, j, m, and $1 \leq m \leq n(n-1)/2$.

(b) By [CV, Lemma 5.1] and its proof, the set of pure tensors in $\mathbb{P}(W^{\perp})$, that is, $\{a \otimes b \in \mathbb{P}(W^{\perp}) : a, b \in S_1\}$, is in one-to-one correspondence with the zero locus Γ, in $\mathbb{P}(S_1) \times \mathbb{P}(S_1)$, of W given by $\Gamma = \{(a,\, b) \in \mathbb{P}(S_1) \times \mathbb{P}(S_1) : w(a,\, b) = 0 \text{ for all } w \in W\}$.

We will now make more precise the connection between points in the zero locus of W and certain quadratic forms.

Lemma 1.8. *If $a, b \in S_1^{\times}$, then the quadratic form $ab \in \mathbb{P}\big(\sum_{i=1}^{n} \Bbbk q_i\big)$ if and only if $(a,\, b) \in \Gamma$.*

Proof. Suppose $w(a,\, b) = 0$ for all $w \in W$. By Remark 1.7(b), $w(a \otimes b) = 0$ for all $w \in W$, and so $a \otimes b \in W^{\perp}$. Since S is a domain, $ab \neq 0$ in S, so $ab \in \mathbb{P}\big(\sum_{i=1}^{n} \Bbbk q_i\big)$, as desired. This argument is reversible, so the converse holds. ■

2. Point Modules over Graded Skew Clifford Algebras

In this section, we prove results that relate point modules over GSCAs to noncommutative quadrics in the sense of Definitions 1.1 and 1.5. In particular, we use the notion of μ-rank introduced in [VV] to extend results in [VVW] about graded Clifford algebras (GCAs) to GSCAs, with our main result being Theorem 2.12. Although the overall approach and some of the proofs are influenced by those in [VVW, §1], many of the proofs involve new arguments.

In [VV], a notion of μ-rank of a (noncommutative) quadratic form on n generators was defined, where $n = 2$ or 3. The results in [VV] suggest a notion of μ-rank at most two of a (noncommutative) quadratic form on n generators for any $n \in \mathbb{N}$ as follows.

Definition 2.1. [VV] Let S be as in the paragraph preceding Definition 1.1, where n is an arbitrary positive integer, and let $Q \in S_2$.

(a) If $Q = 0$, we define μ-rank$(Q) = 0$.
(b) If $Q = L^2$ for some $L \in S_1^{\times}$, we define μ-rank$(Q) = 1$.
(c) If $Q \neq L^2$ for any $L \in S_1^{\times}$, but $Q = L_1 L_2$ where $L_1,\, L_2 \in S_1^{\times}$, we define μ-rank$(Q) = 2$.

Moreover, if $M \in \mathbb{P}(M^{\mu}(n, \Bbbk))$ and if μ-rank$(\tau(M)) \leq 2$, where τ is given in Notation 1.2, then we define μ-rank(M) to be the μ-rank of $\tau(M)$.

Remark 2.2. In contrast to the commutative setting, there exist noncommutative quadratic forms q where $0 \neq q = L^2 = L_1 L_2$, with $L, L_1, L_2 \in S_1$ and $L_1,\, L_2$ linearly independent. For example, let $n = 2 = \mu_{12}$ and $q = (z_1 + 2z_2)^2 = (z_1 + z_2)(z_1 + 4z_2)$.

We now define a function Φ that will play a role similar to that played by the function ϕ in [VVW, §1].

Definition 2.3. Let $a,\, b \in \mathbb{P}^{n-1}$, with $a = (a_1, \dots, a_n)$, $b = (b_1, \dots, b_n)$, where $a_i,\, b_i \in \Bbbk$ for all i. Since $a_k b_\ell \neq 0$ for some $k,\, \ell$, the corresponding 2×2 block in the matrix $(a_i b_j + \mu_{ij} a_j b_i)$ is nonzero, since either $a_k b_k \neq 0$, or $a_\ell b_\ell \neq 0$, or $a_k b_\ell + \mu_{k\ell} a_\ell b_k \neq 0$, so the matrix $(a_i b_j + \mu_{ij} a_j b_i)$ is nonzero and μ-symmetric and defined up to a nonzero scalar multiple. Thus, we may define $\Phi \,:\, \mathbb{P}^{n-1} \times \mathbb{P}^{n-1} \to \mathbb{P}(M^\mu(n, \Bbbk))$ by

$$(a,\, b) \mapsto (a_i b_j + \mu_{ij} a_j b_i) \quad \text{for all } i,\, j = 1, \dots, n.$$

Remark 2.4. With $a,\, b$ as in Definition 2.3, let $q \in S_2$ be the quadratic form

$$q = \left(\sum_{i=1}^n a_i z_i \right) \left(\sum_{i=1}^n b_i z_i \right) \in \mathbb{P}(S_2),$$

so μ-rank$(q) \leq 2$. However, using the relations of S, we find

$$q = \sum_{i=1}^n a_i b_i z_i^2 + \sum_{\substack{i,j=1 \\ i<j}}^n (a_i b_j + \mu_{ij} a_j b_i)\, z_i z_j.$$

It follows that $q = \tau(M)$, where $M = (a_i b_j + \mu_{ij} a_j b_i)$, so $M \in \Bbbk^\times \Phi(a,\, b)$. Hence, μ-rank$(\Phi(a,\, b)) \leq 2$ for all $a,\, b \in \mathbb{P}^{n-1}$.

Proposition 2.5. $\mathrm{Im}(\Phi) = \{M \in \mathbb{P}(M^\mu(n, \Bbbk)) : \mu\text{-rank}(M) \leq 2\}$.

Proof. By Remark 2.4, $\mathrm{Im}(\Phi) \subseteq \{M \in \mathbb{P}(M^\mu(n, \Bbbk)) : \mu\text{-rank}(M) \leq 2\} = X$. Conversely, let $M \in X$ and write $q = \tau(M) \in \mathbb{P}(S_2)$. Since μ-rank$(q) \leq 2$, we have $q = ab$ for some $a,\, b \in S_1^\times$, where $a = \sum_{i=1}^n a_i z_i$ and $b = \sum_{i=1}^n b_i z_i$, with $a_i,\, b_i \in \Bbbk$ for all i. By Remark 2.4, it follows that $M = \Phi((a_i), (b_j))$. ■

Remark 2.6. Recall the notation in Remark 1.7, and suppose $(a,\, b) \in \mathbb{P}(S_1) \times \mathbb{P}(S_1)$. By our assumption in Remark 1.7(a), the point $(a,\, b) \in \Gamma$ if and only if $\sum_{i,j} \alpha_{ijm}(a_i b_j + \mu_{ij} a_j b_i) = 0$ for all m, where $a = (a_i)$, $b = (b_j)$; that is, if and only if the μ-symmetric matrix $\Phi(a,\, b)$ is a zero of $\sum_{i,j} \alpha_{ijm} X_{ij}$ for all m, where X_{ij} is the ij'th coordinate function on $M(n, \Bbbk)$.

Proposition 2.7. *With the assumption in Remark 1.7(a),*

$$\mathrm{Im}(\Phi|_\Gamma) = \left\{ M \in \mathbb{P}\Big(\sum_{k=1}^n \Bbbk M_k \Big) : \mu\text{-rank}(M) \leq 2 \right\}.$$

Proof. Let $H = \{M \in \mathbb{P}\big(\sum_{k=1}^n \Bbbk M_k\big) : \mu\text{-rank}(M) \leq 2\}$ and let $M \in H$. Since M is μ-symmetric of μ-rank at most two, there exists $(a,\, b) \in \mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$ such that $\Phi(a,\, b) = M$, by Proposition 2.5. Thus, by Lemma 1.8, $(a,\, b) \in \Gamma$, so $H \subseteq \mathrm{Im}(\Phi|_\Gamma)$.

For the converse, our argument follows that of [VVW, Proposition 1.5]. Let $(a_{ij}) \in \operatorname{Im}(\Phi|_\Gamma)$. By Proposition 2.5, μ-rank$((a_{ij})) \leq 2$. Setting $Y_{ij} = x_i x_j + \mu_{ij} x_j x_i \in A$, Definition 1.3(a) implies that

$$(Y_{ij}) = \sum_{k=1}^{n} M_k y_k, \tag{1}$$

where equality of the entries is computed in A. By Remark 1.7(a), the defining relations of A are $\sum_{i,j=1}^{n} \alpha_{ijm} Y_{ij} = 0$, $m = 1, \ldots, n(n-1)/2$, and, by Remark 2.6, $\sum_{i,j=1}^{n} \alpha_{ijm} Y_{ij}|_{a_{ij}} = 0$ for all m (where the notation $|_{a_{ij}}$, here and below, denotes evaluation of Y_{ij} at a_{ij} for all i, j). Hence, if f, $g \in \sum_{i,j=1}^{n} \Bbbk Y_{ij}$, then

$$f|_{a_{ij}} = g|_{a_{ij}} \tag{2}$$

whenever $f = g$ in A. Moreover, by Remark 1.4, for each $k \in \{1, \ldots, n\}$, $y_k \in (A_1)^2$; in particular, by Definition 1.3(a), $y_k \in \sum_{i,j=1}^{n} \Bbbk Y_{ij}$. Thus,

$$\begin{aligned} (a_{ij}) = (Y_{ij})|_{a_{ij}} &= \sum_{k=1}^{n} M_k y_k|_{a_{ij}} \qquad \text{by (1) and (2)} \\ &\in \sum_{k=1}^{n} \Bbbk M_k, \end{aligned}$$

since $y_k|_{a_{ij}} \in \Bbbk$ for all k. Hence, $(a_{ij}) \in H$, and so $\operatorname{Im}(\Phi|_\Gamma) \subseteq H$. ■

In order to use Φ to help count the point modules over a regular GSCA, we need to determine which (noncommutative) quadratic forms factor uniquely. To do this, we first prove, in Theorem 2.8, that a quadratic form can be factored in at most two distinct ways.

Theorem 2.8. *A quadratic form can be factored in at most two distinct ways up to a nonzero scalar multiple.*

Proof. Let $q \in S_2^\times$. If q cannot be factored, then the result is trivially true. Hence, we may assume

$$q = \left(\sum_{i=1}^{n} \beta_i z_i \right) \left(\sum_{i=1}^{n} \beta_i' z_i \right),$$

where β_i, $\beta_i' \in \Bbbk$ for all i. If $n = 2$, then the result follows from [VV, Lemma 2.2]. Hereafter, suppose that $n \geq 3$ and that the result holds for $n-1$ generators.

Case I. Suppose $\beta_i \beta_i' \neq 0$ for some i. Without loss of generality, we may assume that $i = n$ and that $\beta_n = 1 = \beta_n'$. Suppose q factors in the following three ways:

$$q = (a + z_n)(a' + z_n) = (b + z_n)(b' + z_n) = (c + z_n)(c' + z_n),$$

where $a, a', b, b', c, c' \in \sum_{k=1}^{n-1} \Bbbk z_k$. Let $\bar{q}$ denote the image of q in $S/\langle z_n \rangle$; clearly, $\bar{q} = aa' = bb' = cc'$. The induction hypothesis implies that $\bar{q}$ factors in at most two distinct ways up to a nonzero scalar multiple. Thus, without loss of generality, we may assume that $c = b$ and $c' = b'$. It follows that q factors in at most two distinct ways up to a nonzero scalar multiple.

Case II. Suppose $\beta_i \beta_i' = 0$ for all i, so $q = \sum_{i<j} \delta_{ij} z_i z_j$ where $\delta_{ij} \in \Bbbk$ for all i, j. We may assume, without loss of generality, that there exists $k \in \{1, \ldots, n\}$ such that $\beta_i = 0$ for all $i > k$ and $\beta_i' = 0$ for all $i \leq k$. By the induction hypothesis, we may also assume that $\beta_i \neq 0$ for all $i \leq k$ and $\beta_i' \neq 0$ for all $i > k$.

If $q \in \langle z_i \rangle$ for some i, we may assume $i = n$ and so $k = n-1$. It follows that $q = az_n = z_n b$, where $a, b \in \sum_{i=1}^{n-1} \Bbbk z_i$. If $q = z_n b'$, where $b' \in S_1$, then $b = b'$ since S is a domain; similarly, if $q = a' z_n$. Moreover, the image of q in the domain $S/\langle z_n \rangle$ is zero, so if also $q = cd$, where $c, d \in S_1$, then $c \in \Bbbk z_n$ or $d \in \Bbbk z_n$ (since $\deg(z_n) = 1$), so q factors in at most two distinct ways up to a nonzero scalar multiple.

Suppose $q \notin \langle z_i \rangle$ for all $i = 1, \ldots, n$, and let $\bar{q}$ denote the image of q in $S/\langle z_n \rangle$. By the induction hypothesis, $\bar{q}$ factors in at most two distinct ways up to a nonzero scalar multiple, so we may assume $\bar{q} = ab = cd$, where $c, d \in \sum_{i=1}^{n-1} \Bbbk z_i$ and $a = \sum_{i=1}^{k} \beta_i z_i$ and $b = \sum_{i=k+1}^{n-1} \beta_i' z_i$. Lifting to S, we have

$$q = a(b + \beta_n' z_n) \quad \text{and} \quad q = c(d + \alpha z_n) \text{ or } (c + \gamma z_n)d,$$

where $\alpha, \gamma \in \Bbbk^\times$, and these are the only ways q can factor in S. Hence, if q factors in three distinct ways in S, then $\beta_n' a z_n = \alpha c z_n = \gamma z_n d$, since $ab = cd$. It follows that $c = \beta_n' \alpha^{-1} a$, since S is a domain, and $b = \beta_n' \alpha^{-1} d$, since $S/\langle z_n \rangle$ is a domain, and so $a(b + \beta_n' z_n)$ is a nonzero scalar multiple of $c(d + \alpha z_n)$ and γ has a unique solution. Thus, q factors in at most two distinct ways up to a nonzero scalar multiple. ■

Theorem 2.8 brings us close to our goal of generalizing (most of) [VVW, Theorem 1.7] from the setting of GCAs to the setting of GSCAs. We first require one last technical result.

Lemma 2.9. *Let Δ_μ denote the points $(a, b) \in \mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$ such that $(\tau \circ \Phi)(a, b)$ factors uniquely (up to nonzero scalar multiple). The restriction of $\tau \circ \Phi$ to $(\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}) \setminus \Delta_\mu$ has degree two and is unramified, whereas $\tau \circ \Phi|_{\Delta_\mu}$ is one-to-one.*

Proof. The result is an immediate consequence of Theorem 2.8 and the definition of Δ_μ. ■

Our next result generalizes (most of) [VVW, Theorem 1.7], which we now state for comparison. A point module is a base-point module with Hilbert series $H(t) = 1/(1-t)$.

Theorem 2.10. [VVW, Theorem 1.7] *Let C denote a GCA determined by symmetric matrices $N_1, \ldots, N_n \in M(n, \Bbbk)$ and let $\mathcal{Q}$ denote the corresponding (commutative) quadric system* *in $\mathbb{P}^{n-1}$. If $\mathcal{Q}$ has no base points, then the number of isomorphism classes of left (respectively, right) point modules over C is equal to $2r_2 + r_1 \in \mathbb{N} \cup \{0, \infty\}$, where r_j denotes the number of matrices in $\mathbb{P}\big(\sum_{k=1}^{n} \Bbbk N_k\big)$ that have rank j. If the number of left (respectively, right) point modules is finite, then $r_1 \in \{0, 1\}$.*

Remark 2.11. In the setting of GCAs, if M is a symmetric matrix, then $\tau(M)$ is a commutative quadratic form where S, in this case, is commutative; thus, if a, $b \in S_1^{\times}$ are linearly independent, then we view $q = ab = ba$ as two different ways to factor q in S. It follows that a symmetric matrix M has rank j, where $j = 1$ or 2, if and only if $\tau(M)$ factors in j distinct ways, up to a nonzero scalar multiple. With this in mind, our next result is clearly a generalization of the first part of Theorem 2.10.

Theorem 2.12. *If the quadric system $\{q_1, \dots, q_n\}$ associated to the GSCA, A, is normalizing and base-point free, then the number of isomorphism classes of left (respectively, right) point modules over A is equal to $2f_2 + f_1 \in \mathbb{N} \cup \{0, \infty\}$, where f_j denotes the number of matrices M in $\mathbb{P}\big(\sum_{k=1}^{n} \Bbbk M_k\big)$ such that μ-rank$(M) \leq 2$ and such that $\tau(M)$ factors in j distinct ways (up to a nonzero scalar multiple).*

Proof. By [ATV1, CV, CVc], the hypotheses on A imply that the set of isomorphism classes of left (respectively, right) point modules over A is in bijection with Γ. Hence, the result follows from Lemma 1.8, Proposition 2.7 and Lemma 2.9. ■

The last part of Theorem 2.10 appears not to extend to the setting of GSCAs. More precisely, the proof of the last part of Theorem 2.10 uses the correspondence between rank and factoring described in Remark 2.11. Given Remark 2.2, the obvious counterpart in the setting of GSCAs is either $f_1 \in \{0, 1\}$ or the number of elements of μ-rank one being at most one. However, the following two examples demonstrate that neither of these properties alone is suitable for generalizing the last part of Theorem 2.10 to the setting of GSCAs.

Example 2.13. Take $n = 4$ and let

$$\mu_{12} = \mu_{13} = \mu_{14} = -\mu_{23} = \mu_{24} = \mu_{34} = 1,$$
$$q_1 = z_4^2, \qquad q_2 = z_2 z_3, \qquad q_3 = (z_1 + z_2)(z_1 + z_4),$$
$$q_4 = b^2 z_1^2 - a^2 z_2^2 + z_3^2 + 2b z_1 z_3,$$

where a, $b \in \Bbbk^{\times}$ and $a^2 \neq b^2$. Since the quadric system is normalizing and base-point free, the corresponding GSCA, A, is quadratic and regular of global dimension four (by Theorem 1.6), and is the $\Bbbk$-algebra on generators $x_1, \dots, x_4$ with defining relations:

$$\begin{aligned} x_1x_2 + x_2x_1 &= x_1^2 - b^2x_3^2, & x_1x_3 + x_3x_1 &= 2bx_3^2, \\ x_1x_4 + x_4x_1 &= x_1^2 - b^2x_3^2, & x_3x_4 + x_4x_3 &= 0, \\ x_2x_4 + x_4x_2 &= x_1^2 - b^2x_3^2, & x_2^2 + a^2x_3^2 &= 0, \end{aligned}$$

and has exactly eleven point modules. In this example, A is a GCA, but the algebra S has been chosen to be noncommutative (via the choice of μ_{23}). Here, $\mathbb{P}\big(\sum_{k=1}^{4} \Bbbk q_k\big)$ contains three elements that factor uniquely, namely

$$q_1, \qquad q_4 + 2aq_2 \qquad \text{and} \qquad q_4 - 2aq_2.$$

(To see that q_4+2aq_2 factors uniquely, we note that the only way it can factor is as $q_4+2aq_2 = (bz_1 + \alpha z_2 + z_3)(bz_1 + \beta z_2 + z_3)$, for some α, $\beta \in \Bbbk$, since its image factors uniquely in $S/\langle z_2\rangle$; solving for α, β yields only one solution: $\alpha = a$, $\beta = -a$. Similarly, for $q_4 - 2aq_2$.) Hence, A has a finite number of point modules, yet $f_1 \geq 3 > 1$.

In the previous example, if, instead, one takes $\mu_{23} = 1$, so that S is now commutative (as in [VVW]), then the quadric system contains only one element of rank one (up to nonzero scalar multiple), which agrees with Theorem 2.10.

Example 2.14. For our second example, we consider a GSCA in [CV, §5.3] with $n = 4$, where

$$q_1 = z_1z_2, \qquad q_2 = z_3^2, \qquad q_3 = z_1^2 - z_2z_4, \qquad q_4 = z_2^2 + z_4^2 - z_2z_3,$$
$$\mu_{23} = 1 = -\mu_{34}, \qquad (\mu_{14})^2 = \mu_{24} = -1, \qquad \mu_{13} = -\mu_{14},$$

so the quadric system is normalizing and base-point free. By Theorem 1.6, the corresponding GSCA, A, is quadratic and regular of global dimension four, and is the $\Bbbk$-algebra on generators $x_1, \ldots, x_4$ with defining relations:

$$\begin{aligned} x_1x_3 &= \mu_{14}x_3x_1, & x_3x_4 &= x_4x_3, & x_2x_3 + x_3x_2 &= -x_4^2, \\ x_1x_4 &= -\mu_{14}x_4x_1, & x_4^2 &= x_2^2, & x_2x_4 - x_4x_2 &= -x_1^2, \end{aligned}$$

and has exactly five nonisomorphic point modules, two of which correspond to $q_1 = z_1z_2 = z_2z_1$. The other three point modules correspond to two quadratic forms in $\mathbb{P}\big(\sum_{k=1}^{4} \Bbbk q_k\big)$ that have μ-rank one, namely

$$q_2 = z_3^2 \qquad \text{and} \qquad q_2 + 4q_4 = (z_2 - \frac{z_3}{2} + z_4)^2 = (-z_2 + \frac{z_3}{2} + z_4)^2,$$

where the latter quadratic form clearly factors in two distinct ways. Hence, A has a finite number of point modules even though two distinct elements of $\mathbb{P}\big(\sum_{k=1}^{4} \Bbbk q_k\big)$ have μ-rank one.

Under the hypotheses of Theorem 2.12, if $\mathbb{P}\big(\sum_{k=1}^{n} \Bbbk q_k\big)$ contains distinct elements a^2 and b^2, where a, $b \in S_1$ are normal in S, such that a^2 and b^2 both factor uniquely in S, then

$\mathbb{P}\big(\sum_{k=1}^{n} \Bbbk q_k\big)$ contains infinitely many elements of μ-rank two, and so, in this special case, the corresponding regular GSCA has infinitely many point modules. However, in general, if c^2 factors uniquely in S, where $c \in S_1^{\times}$, then c need not be normal in S, as can be seen by taking $1 = \mu_{12} \neq \mu_{13}\mu_{32}$ and $c = z_1 + z_2$.

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