EMBEDDING A QUANTUM NONSINGULAR QUADRIC IN A QUANTUM $\mathbb{P}^3$

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Abstract. A definition of regularity was given in [2] for non-commutative graded algebras and the results of [2] together with those in [4, 5] classify the regular algebras of global dimension three that are generated by degree one elements. Our purpose is to classify a certain class of quadratic regular algebras of global dimension four.

Let S be a twisted homogeneous coordinate ring of a nonsingular quadric $Q \subset \mathbb{P}^3$. Our interest is in algebras R such that (in the language of [1]) there is an embedding Proj $S \hookrightarrow$ Proj R. In this paper, we classify all the quadratic regular algebras R of global dimension four which have the same Hilbert series as that of the polynomial ring on four variables, and which map onto S via a graded degree zero homomorphism. Our approach follows [4, 5, 8], making use of the point modules of R and their associated geometric data. We classify the algebras R according to their "point scheme" P and corresponding automorphism $\sigma \in \mathrm{Aut}(P)$; those algebras R which are determined by (P, σ) belong to at most a five-parameter family, but those which are not determined by (P, σ) belong to at most a four-parameter family. In the first case, P is either $\mathbb{P}^3$ or consists of Q together with a line L, whilst in the second case $P = Q$.

It is also proved that under certain sufficient conditions, the zero locus of the defining relations of a quadratic regular algebra of global dimension four is the graph of an automorphism.

Introduction

A notion of non-commutative regular (graded) algebra was introduced in [2], and a classification of the regular algebras of global dimension three was begun. It was completed for algebras generated by degree one elements in [4, 5] by using mainly geometric techniques – the key idea being to associate certain graded modules to geometric data, such as "point modules" to points, "line modules" to lines, etc. To further these techniques, Artin defined a notion of Proj A in [1] for a non-commutative graded algebra A generated by degree one elements. Regular algebras of global dimension four have not been classified as the problem

1991 *Mathematics Subject Classification.* 16W50, 16P40, 14A22.

Key words and phrases. regular algebra, quadratic algebra, point scheme, point variety, point module.

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is much harder than for dimension three. Although several such examples were studied in [7, 8, 9, 11, 12, 13, 15, 16, 19, 20, 21] using similar geometric techniques to [4, 5], it is still unknown which regular algebras of global dimension four are generic or exceptional, nor the types of behaviour that are to be expected. The purpose of this paper is to classify a certain class of quadratic regular algebras of global dimension four.

Let S denote a twisted homogeneous coordinate ring of a nonsingular quadric $Q \subset \mathbb{P}^3$ corresponding to an automorphism τ of Q ([19, §3]). Our interest is in algebas R such that Proj $S \hookrightarrow$ Proj R – the objective being to classify the quadratic regular algebras R of global dimension four which map onto S via a graded degree zero homomorphism such that the Hilbert series of R is the same as that of the polynomial ring on four variables. We classify such R in Theorems 2.14 and 4.13 according to the scheme P determined by the point modules of R (also see Theorems 2.6 and 2.9 and Propositions 2.7 and 2.10).

In [19, 21] examples are given of such algebras R. In [19] the span of the defining relations of the algebras gives the bihomogeneous forms which vanish on the graph of an automorphism σ of $Q \cup L$ where L is a line in general position and $\sigma|_Q = \tau$. Many algebraic properties of those algebras depend on the geometric data $\{Q \cup L, \sigma\}$; in particular, the point modules are parametrised by $Q \cup L$ so, in the terminology of [8], the "point variety" of such an algebra is $Q \cup L$. On the other hand, in [21], τ was assumed to preserve the rulings on Q, and Schelter's program "Affine" was used to find quadratic regular algebras R whose defining relations have zero locus which contains Q. In this way, an algebra $\widetilde{R}$ was found whose defining relations vanish on the graph $\Gamma_\tau(Q)$ of τ but the bihomogeneous forms that vanish on $\Gamma_\tau(Q)$ give the defining relations of S. This behaviour differs from previously studied examples of quadratic regular algebras in that the geometric data associated to $\widetilde{R}$ determines S and not $\widetilde{R}$. Indeed, it follows from Theorem 4.13 that corresponding to each S (that is, to each (Q, τ)) there is a one-parameter family of such regular algebras $\widetilde{R}$ (up to isomorphism).

In [16] Stafford undertook a similar project for the twisted homogeneous coordinate ring B of an elliptic curve in $\mathbb{P}^3$ which is associated to the Sklyanin algebra of global dimension four ([13, 6]); that is, he classified all the quadratic regular algebras of global dimension four which map onto B (via a regular normalizing sequence) and have the same Hilbert series as that of the polynomial ring on four variables. Stafford described the algebras as being "not determined by their geometric data", by which he means the algebras are not determined by the ring B. However, the point variety of his algebras was different from that of B, so in our terminology (Definition 1.13) his algebras *are* determined by their geometric data, by which we mean their point variety (or "point scheme") and corresponding automorphism.

One of our main results is Theorem 1.10 which gives sufficient conditions for the zero locus of the defining relations of a regular algebra of global dimension four to be the graph of an automorphism. Virtually all of the quadratic regular algebras studied to date satisfy those sufficient conditions.

In our analysis it is important to distinguish between the scheme and the variety determined by the point modules; in [8] the latter is called the "point variety" but the former is used without name. In view of Theorem 1.10, we introduce the terminology "point scheme" $(\mathcal{P}, \sigma)$ of a quadratic regular algebra, where $\mathcal{P}$ is the scheme determined by the point modules and $\sigma \in \operatorname{Aut}(\mathcal{P})$ is the shift functor on point modules. In our setting, the point scheme (P, σ) is determined by the zero locus of the defining relations of R and splits the analysis of R into two cases: either R is determined by its geometric data (P, σ), or it is not. In either case, $Q \subset P$ and $\sigma|_Q = \tau$, but in the first case $P \neq Q$, whereas in the second, $P = Q$. The first case is the simpler one since there the geometry is a major tool, whereas the second case requires additional techniques since the point scheme no longer encodes sufficient data on R.

The structure of the paper is as follows. There are four sections, the first of which gives definitions and proves some general results including Theorem 1.10 mentioned above. Section 2 focuses on the case $P \neq Q$ which corresponds to those R which are determined by their geometric data, whilst §4 is devoted to the case $P = Q$ which corresponds to those R which are not determined by their geometric data. Section 3 introduces techniques to be used in §4, but applies to all $P \neq \mathbb{P}^3$. Our approach to the case $P = Q$ (used in §§3, 4), which uses the Koszul duals of R and S, was motivated by Stafford's approach in [16]. We now give the strategy used in §§2, 4, while summarising differences in the results describing the behaviour of R.

In §2 we first describe the schemes P that arise if $P \neq Q$, and show in Proposition 2.7 that there are three kinds: either $P = \mathbb{P}^3$, or $P = Q \cup L$ where $L \not\subset Q$ is a line that intersects Q at two points (counted with multiplicity), or P is the quadric Q together with an embedded line $L \subset Q$ of multiplicity two (denoted $P = Q \uplus L$) – in the latter case, the reduced variety of P is Q. For all such R, we have $\tau \in \operatorname{Aut}(R)$, and if $P = Q \cup L$ or if $P = Q \uplus L$, then there is a regular normalizing sequence of degree one elements v_1, v_2 such that $L = \mathcal{V}(v_1, v_2)$. It follows that R is a twist by τ of a regular algebra that maps onto the (commutative) homogeneous coordinate ring of Q (Theorem 2.6) and that R is a twist of a regular, central extension of a regular algebra of global dimension three (Theorem 2.9). We classify such R in Theorem 2.14; if $P = Q \cup L$, then R belongs to a three-parameter family, whereas if $P = Q \uplus L$, then R belongs to at most a five-parameter family (up to isomorphism). Examples of such R where $Q \cap L$ consists of two distinct points are analysed in [19], and examples where $P = Q \uplus L$ or where $P = Q \cup L$ with L tangential to Q are given in [21]. Such algebras include some examples from quantum groups (such as the coordinate ring of quantum 2×2 matrices) which are discussed in Example 2.8.

We consider only the case $P = Q$ in §4. Section 4.1 demonstrates that a procedure similar to that used in §2 to classify R fails if $P = Q$; in fact, in contrast to §2, we have $\tau \notin \mathrm{Aut}(R)$, there are no normal elements of degree one, and R is not a twist of a regular central extension of a regular algebra of global dimension three. The purpose of §4.2 is to give necessary conditions on τ, so that, combined with §3, we are able to define a class of algebras $A(\alpha, \tau)$, where $\alpha \in k$, to which R belongs, but conceivably not all $A(\alpha, \tau)$ need be regular. Examples of such R were found where $\alpha^2 = 1$ in [21] using Schelter's program Affine. In §4.3 we use twisting systems (which were defined in [23]) to prove that if α and τ satisfy certain necessary conditions, then $A(\alpha, \tau)$ twists by a twisting system to one particular algebra $A(\alpha, t)$ (Proposition 4.12). It follows that if one regular algebra R exists (where $P = Q$ and α is arbitrary), then all such $A(\alpha, \tau)$ are regular with the desired Hilbert series. The classification of all such R, where $P = Q$, is given in Theorem 4.13 and shows that R belongs to at most a four-parameter family (up to isomorphism).

At the end of §4 we remark on differences between the line modules over R where $P = Q$ versus $P \neq Q$. In particular, in both cases there is a one-to-one correspondence between the left and the right line modules over R, but if $P = Q$, then the set of lines giving the left line modules is different from that giving the right line modules, whereas if $P \neq Q$, the lines coincide. This suggests that if $P = Q$, then the line modules (and corresponding automorphism) may determine R.

1. The Regular Algebra R and its Point Scheme

In §1.1 the regular algebras R which are to be classified are defined and it is shown that R contains a homogeneous element Ω of degree two which is determined by certain geometric data. In Lemma 1.5 we prove that Ω is normal in R. In §1.2 the point scheme (P, σ) of R is defined and Theorem 1.10, which is a general result on regular algebras of global dimension four, is proved. This result is used to show that the zero locus of the defining relations of R in $\mathbb{P}^3 \times \mathbb{P}^3$ is the graph of an automorphism σ of P.

Throughout, k denotes an algebraically closed field such that $\mathrm{char}(k) \neq 2$. In order to define the algebra R, we first review the definitions of regularity and twisting of an algebra by an automorphism.

Definition 1.1. [2] Let A be a connected graded algebra. Following [2], A is called regular (of dimension d) if

(a) A has finite global homological dimension, $\mathrm{gldim}(A) = d$,

(b) A has finite GK-dimension,

(c) A is Gorenstein; that is, $\mathrm{Ext}^q_A(k, A) = \delta^q_d k$.

Occasionally we use the terms Auslander-regular, Auslander-Gorenstein and Cohen-Macaulay property which are defined in [10, 11].

Definition 1.2. [5] Let $A = \bigoplus_{n\geq 0} A_n$ be a quadratic algebra and let ϕ be a graded degree zero automorphism of A. The twist A^ϕ of A by ϕ is the vector space $\bigoplus_{n\geq 0} A_n$ with a new multiplication $*$ defined as follows: if $a, b \in A_1$ then $a * b = ab^\phi$ where the right hand side is computed using the original multiplication in A.

Twisting systems were defined in [23] (see Definition 4.10) and generalize the notion of a twist by an automorphism. Twisting systems define an equivalence relation on the set of associative graded multiplications defined on $\bigoplus_{n\geq 0} A_n$, and determine an isomorphism between the category of graded A-modules and the category of graded B-modules, where B is a twist of A via a twisting system (or an automorphism); in particular, $\operatorname{Proj} A \cong \operatorname{Proj} B$ (see Definition 1.8).

1.1. Definition of the Regular Algebra R.

Let Q denote a nonsingular quadric in $\mathbb{P}^3$ and let $S(Q)$ (respectively, $S(\mathbb{P}^3)$) denote the (commutative) homogeneous coordinate ring of Q (respectively, $\mathbb{P}^3$). Fix an automorphism τ of Q. Since τ is the restriction of a unique automorphism of $\mathbb{P}^3$ (which we also denote by τ), we have that τ is defined on $S(Q)_1$ via $\tau(x) = x \circ \tau$ for all $x \in S(Q)_1$, and τ may be extended to an automorphism of $S(Q)$ in the obvious way. We write x^τ for $\tau(x)$. Let S denote the twist of $S(Q)$ by τ. Then S is the twisted homogeneous coordinate ring $B(Q, \tau, \mathcal{O}_Q(1))$ ([19, §3]). Moreover, any twist of $S(Q)$ (respectively, $S(\mathbb{P}^3)$) by a twisting system is isomorphic to a twist of $S(Q)$ (respectively, $S(\mathbb{P}^3)$) by an automorphism.

Throughout the paper, R denotes a quadratic, regular algebra of global dimension four with Hilbert series $H_R(t) = (1-t)^{-4}$ such that there is a graded, degree zero, onto homomorphism $R \twoheadrightarrow S$. (In particular, in the language of [1, 21], $\operatorname{Proj} S$ embeds in the quantum space $\operatorname{Proj} R$.) Our goal is to classify such algebras R.

The algebra S is quadratic so its defining relations are bihomogeneous forms which vanish on the graph of τ. Since R is quadratic and $H_S(t) = (1+t)(1-t)^{-3}$, there exists $\Omega \in R_2$ such that $S \cong R/\langle\Omega\rangle$. There are no nonzero elements of S_2 that vanish on the graph of τ, so any nonzero element of R_2 that vanishes on the graph of τ is a scalar multiple of Ω. In particular, if $u, v \in R_1$, then $u^\tau v - v^\tau u \in R_2$ and is a scalar (possibly zero) multiple of Ω.

All the algebras R which were found in [19, 21] had the property that Ω was a normal element of R. The following result shows that this is independent of the regularity and Hilbert series hypotheses.

Definition 1.3. Let F be a finite dimensional k-vector space and let $A = T(F)/\langle G\rangle$ where $G \subset F \otimes F$. The Koszul dual of A is defined to be the quadratic algebra $A^! = T(F^*)/\langle G^\perp\rangle$.

Definition 1.4. [8] A homogeneous element a of a graded algebra A is called n-regular if $\dim(aA_m) = \dim(A_m a) = \dim(A_m)$ for all $m \leq n$.

Lemma 1.5. *Let A be a quadratic algebra. If there exists $\Omega \in A_2$ such that $A/\langle\Omega\rangle \cong S$, then Ω is normal.*

Proof. We write $A = T(F)/\langle G\rangle$, where $\dim F = 4$ and $G \subset F \otimes F$ and write $S = T(F)/\langle G + k\tilde{\Omega}\rangle$ where $\tilde{\Omega}$ is a preimage of Ω in $F \otimes F$. Pick $\tilde{\omega} \in G^{\perp}$ such that $\tilde{\omega} \cdot \tilde{\Omega} = 1$ and let ω denote the image of $\tilde{\omega}$ in $S^!_2$. By [16, Lemma 2.5], Ω is normal if and only if ω is 1-regular. The algebra $S^!$ is a twist by an automorphism of $S(Q)^!$, and ω is 1-regular if and only if the image of ω in $S(Q)^!$ is 1-regular. By [16, Lemma 2.5], this holds if and only if the desired result holds in the case $S = S(Q)$ (that is, $\tau =$ identity). Thus we may assume $S = S(Q)$, so $uv - vu \in k\Omega$ for all $u, v \in A_1$.

If $uv = vu$ for every line $\mathcal{V}(u, v) \subset Q$ where $u, v \in A_1$, then $A = S(\mathbb{P}^3)$, so Ω is central. Hence we may assume there exist $x_1, x_2 \in A_1$ such that $\mathcal{V}(x_1, x_2) \subset Q$ and $x_1x_2 - x_2x_1 = \Omega$. Since $\mathcal{V}(x_1, x_2) \subset Q$ we have $\dim(x_1S_1 + x_2S_1) = 6$ and so $\dim(x_1A_1 \cap x_2A_1) = 1$. Thus there exist $x_3, x_4 \in A_1$ such that $x_1x_4 = x_2x_3$ in A. Since $x_1x_2 \neq x_2x_1$ we have $x_4 \notin kx_2$ and $x_3 \notin kx_1$, and since $x_1x_4 = x_2x_3$ in S also, it follows that $Q = \mathcal{V}(x_1x_4 - x_2x_3)$ and $\{x_1, x_2, x_3, x_4\}$ is a linearly independent set.

It follows that $\mathcal{V}(x_3, x_4) \subset Q$ and $\dim(S_1x_3 + S_1x_4) = 6$ so that $\dim(A_1x_3 \cap A_1x_4) = 1$, whence in A we have $x_3x_4 - x_4x_3 \in k^{\times}\Omega$. Moreover, $x_3x_2 - x_4x_1 \in k\Omega$. Hence

$$
\begin{aligned}
x_1\Omega \in k^{\times}x_1(x_3x_4 - x_4x_3) &\subseteq k^{\times}(x_3x_1x_4 - x_2x_3^2) + k\Omega x_4 \\
&= k^{\times}(x_3x_2 - x_2x_3)x_3 + k\Omega x_4 \\
&\subseteq \Omega A_1 \\
x_2\Omega \in k^{\times}x_2(x_3x_4 - x_4x_3) &\subseteq k^{\times}(x_1x_4^2 - x_4x_2x_3) + k\Omega x_3 \\
&= k^{\times}(x_1x_4 - x_4x_1)x_4 + k\Omega x_3 \\
&\subseteq \Omega A_1 \\
x_3\Omega = x_3(x_1x_2 - x_2x_1) &\subseteq k^{\times}(x_1x_3x_2 - x_4x_1^2) + k\Omega x_2 + k\Omega x_1 \\
&\subseteq k^{\times}(x_1x_4 - x_4x_1)x_1 + kx_1\Omega + k\Omega x_2 + k\Omega x_1 \\
&\subseteq \Omega A_1 \\
x_4\Omega = x_4(x_1x_2 - x_2x_1) &\subseteq k^{\times}(x_3x_2^2 - x_2x_4x_1) + k\Omega x_2 + k\Omega x_1 \\
&\subseteq k^{\times}(x_3x_2 - x_2x_3)x_2 + kx_2\Omega + k\Omega x_2 + k\Omega x_1 \\
&\subseteq \Omega A_1.
\end{aligned}
$$

Hence $A_1\Omega \subseteq \Omega A_1$ and similarly $\Omega A_1 \subseteq A_1\Omega$. The result follows. ■

Corollary 1.6. *The algebra R is a noetherian domain, is Auslander-regular of global dimension four, and satisfies the Cohen-Macaulay property.*

Proof. Since S is noetherian, so is R by Lemma 1.5 and [4, Lemma 8.2]. By [5, Theorem 3.9] it follows that R is a domain. Moreover, since Q is a complete intersection and S is a twist of $S(Q)$ by an automorphism, S is Auslander-Gorenstein and satisfies the Cohen-Macaulay property. By [10] and Lemma 1.5, these properties carry up to R. By definition, R has global dimension four, which completes the proof. ∎

If A satisfies the hypotheses of Lemma 1.5, then Ω may be far from regular; indeed, $H_A(t) = H_S(t) + t^2$ is even possible, as the following example demonstrates.

Example 1.7. Let $[a, b]$ denote $ab - ba$, and let $A = k[x_1, \dots, x_4]$ with defining relations $x_1x_4 = x_2x_3$ and $[x_i, x_j] = [x_1, x_2]$ for all $i < j$. Then $A/\langle \Omega = x_1x_2 - x_2x_1 \rangle = S(Q)$ and the point variety of A is Q (with no multiple points) but $\Omega x = 0 = x\Omega$ for all $x \in A_n$ for all $n \in \mathbb{N}$.

1.2. **The Point Scheme.**

The main purpose of this subsection (Theorem 1.10) is to show that the zero locus of the defining relations of R in $\mathbb{P}^3 \times \mathbb{P}^3$ is the graph of an automorphism of some subscheme of $\mathbb{P}^3$.

Definition 1.8. [4] Let A be a graded algebra which is generated by A_1.

(a) A point (respectively, line or plane) module over A is a graded cyclic module M with Hilbert series $H_M(t) = (1 - t)^{-1}$ (respectively, $(1 - t)^{-2}$ or $(1 - t)^{-3}$).

(b) A truncated point module of length n over A is a graded cyclic module M with Hilbert series $H_M(t) = 1 + t + \cdots + t^{n-1}$.

(c) Let $M = \bigoplus M_i$ be a graded module. For $r \in \mathbb{Z}$, the shift $M[r]$ of M is defined by setting $M[r]_i = M_{r+i}$. If M is a point module, then $M[r]$ is generated by its degree $-r$ part. Let $M_{\geq j}$ denote $\bigoplus_{i \geq j} M_i$.

(d) Assume, in addition, that A is noetherian. Following Artin [1] we define the quantum space Proj A of A to be the quotient category of the category of all finitely generated graded left (respectively, right) A-modules modulo the subcategory of finite length modules.

For the rest of this subsection A denotes a quadratic noetherian algebra satisfying the following conditions:

(1) the Hilbert series of A is $H_A(t) = (1-t)^{-4}$, which is the same as that of the polynomial algebra on four variables;

(2) A is Auslander-regular of global dimension four;

(3) A satisfies the Cohen-Macaulay property.

B. Shelton and J. Zhang have pointed out to us that if a graded algebra A is noetherian and regular and has Hilbert series $H_A(t) = (1 - t)^{-4}$, then A is Koszul (so quadratic) and has global dimension four.

The algebras R to be classified in this paper satisfy these conditions and those of the following two results.

Lemma 1.9.
Suppose A satisfies conditions 1-3 above (page 7). For $i = 1, 2$, let $\pi_i \colon \mathbb{P}(A_1^*) \times \mathbb{P}(A_1^*) \to \mathbb{P}(A_1^*)$ denote the projection map onto the i'th coordinate and let W denote the subspace of $A_1 \otimes A_1$ generated by the defining relations of A. Suppose $\pi_i(\mathcal{V}(W))$ contains two distinct points for both $i = 1$ and $i = 2$. If $u, v, w \in A_1$ are linearly independent elements such that dim$(A_1u + A_1v + A_1w) \leq 8$, then there exist linearly independent elements $u', v' \in ku + kv + kw$ such that $A_1u' \cap A_1v' \neq 0$.

Proof. Let u, v, w satisfy the conditions of the lemma and let $p = \mathcal{V}(u, v, w) \in \mathbb{P}(A_1^*)$. Since $\dim(A_1u + A_1v + A_1w) \leq 8$, we have that $A/Au + Av + Aw$ maps onto a (nonunique) truncated point module of length 3, so that $p \in \pi_2(\mathcal{V}(W))$. By the hypothesis on $\pi_2(\mathcal{V}(W))$, there exists $(r_1, r_2) \in \mathcal{V}(W)$ with $r_2 \neq p$. Let $u', v' \in A_1$ be linearly independent elements such that $p, r_2 \in \mathcal{V}(u', v')$; in particular, $u', v' \in ku + kv + kw$.

If the conclusion of the lemma were false, then $\dim(A_1u' + A_1v') = 8$ and $A_1w' \subset A_1u' + A_1v'$ for all $w' \in ku + kv + kw$. It would then follow that $A_1w'(r_1, r_2) \subset (A_1u' + A_1v')(r_1, r_2) = 0$ for all $w' \in ku + kv + kw$. On the other hand, since $r_2 \neq p$, there exists $w' \in ku + kv + kw$ with $w'(r_2) \neq 0$. This would imply that $A_1(r_1) = 0$, which is absurd. ■

Theorem 1.10.
Suppose A satisfies conditions 1-3 above (page 7). For $i = 1, 2$, let $\pi_i \colon \mathbb{P}(A_1^*) \times \mathbb{P}(A_1^*) \to \mathbb{P}(A_1^*)$ denote the projection map onto the i'th coordinate and let W denote the subspace of $A_1 \otimes A_1$ generated by the defining relations of A. If $\pi_i(\mathcal{V}(W))$ contains two distinct points for both $i = 1$ and $i = 2$, then

(a) *$\mathcal{V}(W)$ is the graph of an automorphism σ of some subscheme $\mathcal{P}$ of $\mathbb{P}(A_1^*)$;*
(b) *the isomorphism classes of left (respectively, right) point modules over A are in bijection with $\mathcal{V}(W)$, where σ is the shift functor on point modules (that is, if $M(p)$ is the left point module corresponding to $p \in \mathcal{P}$, then $M(p)_{\geq 1}[1] \cong M(\sigma^{-1}(p))$).*

Note . The proof of this theorem makes use of homological results in [11, §1, §2]. The hypotheses throughout [11, §2] include the assumption that the regular algebra contains a regular normalizing sequence consisting of two degree two elements, which determine a factor algebra that is a twisted homogeneous coordinate ring of an elliptic curve in $\mathbb{P}^3$ ([6]). However, the results we use from [11, §2] (namely, 2.1(e), 2.6-2.9) do not use that hypothesis.

Proof. Part (b) follows from (a) and [8, Theorem 4.1.1]. Moreover, by [8, Theorem 4.1.3], to prove (a), it suffices to prove that the reduced variety of $\pi_1(\mathcal{V}(W))$ is isomorphic to that of $\pi_2(\mathcal{V}(W))$. Let $\pi_i(\mathcal{V}(W))_{\text{red}}$ denote these respective reduced varieties. By definition of

$\mathcal{V}(W)$, if $p_2 \in \pi_2(\mathcal{V}(W))_{\text{red}}$, then there exists $p_1 \in \pi_1(\mathcal{V}(W))_{\text{red}}$ such that $(p_1, p_2) \in \mathcal{V}(W)_{\text{red}}$. Thus it remains to prove that p_1 is unique.

We may define a left truncated point module M of length 3 over A by $M = kv_0 \oplus kv_1 \oplus kv_2 = Av_0$ with A-module action determined by $xv_0 = x(p_2)v_1$, $xv_1 = x(p_1)v_2$ and $xv_2 = 0$ for all $x \in A_1$. Writing $p_2 = \mathcal{V}(u, v, w)$ where $u, v, w \in A_1$, we have that $A/Au + Av + Aw$ maps onto M. Since $\dim(A_2) = 10$, it follows that $\dim_k(A_1u + A_1v + A_1w) \leq 9$.

Suppose that p_1 is not unique. Then $A/Au + Av + Aw$ maps onto a truncated point module M' of length 3 where $M' \ncong M$. It follows that $\dim(A_1u + A_1v + A_1w) \leq 8$. Hence, by Lemma 1.9, we may assume that $p_2 = \mathcal{V}(u, v, w)$ where $u, v \in A_1$, $A_1u \cap A_1v \neq 0$ and $\dim_k(A_1u + A_1v + A_1w) \leq 8$.

By [11, Proposition 2.8], the module $L := A/Au + Av$ is a left line module over A and is critical with respect to GK-dimension. We will show that L contains a shifted point module (which contradicts the criticality of L). Since L maps onto M, we may write

$$L = k\tilde{v}_0 \oplus (k\tilde{v}_1 \oplus kf) \oplus (k\tilde{v}_2 \oplus kg \oplus kh) \oplus \cdots$$

where $L_0 = k\tilde{v}_0$, $L_1 = k\tilde{v}_1 \oplus kf$, $L_2 = k\tilde{v}_2 \oplus kg \oplus kh$, $\tilde{v}_i$ is a preimage of v_i in L and $A_1f \subset kg \oplus kh$. In particular, $\text{GKdim}(Af) = 2$ (so $\dim(A_1f) > 0$) since L is critical of GK-dimension two. By construction,

$$\frac{L}{Af} \cong \frac{A}{Au + Av + Aw} \twoheadrightarrow M;$$

and by our assumption that $\dim_k(A_1u + A_1v + A_1w) \leq 8$, we have $\dim A_1f = 1$. It follows that there exist linearly independent elements u_1, u_2, $u_3 \in A_1$ such that there is a degree zero homomorphism $(A/Au_1 + Au_2 + Au_3)[-1] \twoheadrightarrow Af$. In particular, $7 \leq \dim_k(A_1u_1 + A_1u_2 + A_1u_3) \leq 9$. On the other hand, if $A_1u_1' \cap A_1u_2' \neq 0$ for some linearly independent elements u_1', $u_2' \in ku_1 + ku_2 + ku_3$, then $A/Au_1' + Au_2'$ is critical of GK-dimension two giving that $Au_1 + Au_2 + Au_3 \subset Au_1' + Au_2'$, whence $\dim(A_1f) = 2$, which is a contradiction. Hence, by Lemma 1.9, we have $\dim_k(A_1u_1 + A_1u_2 + A_1u_3) = 9$, so $\dim A_2f = 1$. Continuing in this way, we find that the Hilbert series of Af is $H_{Af}(t) = t(1-t)^{-1}$, which contradicts the fact that $\text{GKdim}(Af) = 2$.

It follows that p_1 is unique, and observing the symmetry of the arguments completes the proof. ■

Remark 1.11. With the hypotheses of Theorem 1.10, but not the "two-point hypothesis", it is still true that the point modules are in bijection with the graph of an automorphism of a subscheme of $\mathbb{P}(A_1^*)$, but it is conceivable that the graph is not all of $\mathcal{V}(W)$. In particular, without the extra hypothesis, it is conceivable that there exists a truncated point module of length 3 which cannot be extended to one of length 4.

Henceforth, we write $R = T(V)/\langle W\rangle$ where V is a 4-dimensional k-vector space and W is the subspace of $V \otimes V$ generated by the defining relations of R. As vector spaces, $V \cong R_1 \cong S_1$. By Corollary 1.6 and Theorem 1.10, we have that $\mathcal{V}(W)$ is the graph Γ of an automorphism σ of some subscheme P of $\mathbb{P}^3$. It follows that $Q \subset P$ and $\sigma|_Q = \tau$. We denote the graph of τ on Q by $\Gamma_\tau(Q)$. In the terminology of [8], we call the reduced variety P_{red} of P the "point variety" of R. This is motivated by the following result.

Lemma 1.12. *The left (respectively, right) plane modules over R are in one-to-one correspondence with the planes in $\mathbb{P}^3$. The isomorphism classes of the left (respectively, right) point modules over R are in one-to-one correspondence with P_{red}; if $p \in P_{red}$, then the corresponding left (respectively, right) point module is given by R/I where I is a left (respectively, right) ideal generated by the subspace of R_1 that vanishes at p.*

Proof. The first statement is a consequence of the fact R is a domain, and the second follows from Theorem 1.10 and [8, Theorem 4.1.1]. ■

Suppose A satisfies the hypotheses of Theorem 1.10. We call $(\mathcal{P}, \sigma)$, or simply $\mathcal{P}$, the "point scheme" of A, where $\mathcal{P}$ is the scheme determined by the point modules and σ is the shift functor on point modules. In our setting, the point scheme of R is (P, σ).

Let $\{f_i : i = 1, \dots, 6\}$ be a basis for the span W of the defining relations of R. Then $f_i = \sum_{j=1}^4 m_{ij} \otimes x_j = \sum_{j=1}^4 x_j \otimes m'_{ji}$ in $T(V)$ for some m_{ij} and $m'_{ji} \in V$. The schemes determined by the zeros of the 4×4 minors of the matrix (m_{ij}) and of the matrix (m'_{ji}) are equal by Theorem 1.10, and so determine P. If $f \in T(V)_2$ we say that $f|_{\mathcal{V}(W)} = 0$ if the two schemes determined by the 4×4 minors of the matrices determined by $\{f, f_i : i = 1, \dots, 6\}$ are equal to P. Since S is a proper quotient of R, it follows that if $P \neq Q$, then $\Omega|_{\mathcal{V}(W)} \neq 0$.

Definition 1.13. Let $\mathcal{P}$ be a subscheme of $\mathbb{P}^n$ and $\theta \in \text{Aut}(\mathcal{P})$. We define the quadratic algebra determined by the geometric data $(\mathcal{P}, \theta)$ to be the graded algebra

$$\frac{T(U)}{\langle f \in U \otimes U : f|_{\Gamma_\theta} = 0\rangle}$$

where U is an $(n+1)$-dimensional k-vector space, $T(U)$ is the tensor algebra on U and $\Gamma_\theta \subset \mathbb{P}(U^*) \times \mathbb{P}(U^*)$ is the graph of θ. (Here $\mathbb{P}(U^*)$ is identified with the copy of $\mathbb{P}^n$ containing $\mathcal{P}$.)

The algebra S is the quadratic algebra determined by the geometric data (Q, τ).

An example in [21] shows that the case $P = Q$ may arise. If $P = Q$, then R is not the algebra determined by the geometric data (P, σ) since in this case $\sigma = \tau$ and this data determines S. This is in contrast to the case $P \neq Q$ as the following result demonstrates.

Lemma 1.14. *If the point scheme P is not the quadric Q, then R is determined by the geometric data (P, σ); that is,*

$$R \cong \frac{T(V)}{\langle f \in V \otimes V : f|_{\mathcal{V}(W)} = 0 \rangle}.$$

Proof. Let $f \in R_2$ with $f|_{\mathcal{V}(W)} = 0$. Then f vanishes on $\Gamma_\tau(Q)$ so $f \in k\Omega$. However, $\Omega|_{\mathcal{V}(W)} \neq 0$, so $f = 0$ in R. ■

Corollary 1.15. *The point scheme P of R is $\mathbb{P}^3$ if and only if R is a twist by the automorphism τ of the polynomial algebra $S(\mathbb{P}^3)$.*

Proof. The point scheme of $S(\mathbb{P}^3)$ is $\mathbb{P}^3$, and by [5], twisting by an automorphism preserves the point variety, so sufficiency is proved. Conversely, if $P = \mathbb{P}^3$, then the automorphism σ is linear, so $\sigma = \tau$. It then follows from Lemma 1.14 that $u^\tau v = v^\tau u$ in R for all $u, v \in R_1$, which completes the proof. ■

Lemma 1.16. *The point variety P_{red} is invariant under the automorphism τ, and τ maps multiple points of P corresponding to points on Q to multiple points of P corresponding to points on Q.*

Proof. If $P = Q$, then there is nothing to prove, since $\tau \in \mathrm{Aut}(Q)$. If $P \neq Q$, then τ maps multiple points of P to multiple points of P^τ. However, $\sigma \in \mathrm{Aut}(P)$, so σ maps multiple points of P to multiple points of P, so since $\sigma|_Q = \tau$, the collection of multiple points of P corresponding to points on Q is invariant under τ.

It remains to prove that $(P_{\mathrm{red}} \setminus Q)^\tau = P_{\mathrm{red}} \setminus Q$. By Lemma 1.12, the point modules over R correspond to the points of P_{red}, so we will prove that if $p \in P_{\mathrm{red}} \setminus Q$, then $\tau^{-1}(p)$ corresponds to a point module over R, so that $\tau^{-1}(p) \in P_{\mathrm{red}} \setminus Q$.

Let $M(p)$ denote the left point module corresponding to $p \in P_{\mathrm{red}} \setminus Q$. Then $M(p) \cong R/Ru + Rv + Rw$ for some $u, v, w \in R_1$ such that $p = \mathcal{V}(u, v, w)$. Since p does not correspond to a point module over S, it follows that $\Omega \notin Ru + Rv + Rw$. Hence $0 = u^\tau v - v^\tau u = v^\tau w - w^\tau v = u^\tau w - w^\tau u$. By Corollary 1.6 and [11, Proposition 2.8], $R/u^\tau R + v^\tau R$ is a line module and so $\dim(u^\tau R_1 \cap v^\tau R_1) = 1$. Similarly for (u^τ, w^τ) and (v^τ, w^τ), so $\dim(u^\tau R_1 + v^\tau R_1 + w^\tau R_1) \leq 9$. We will show that $R/u^\tau R + v^\tau R + w^\tau R$ is a right point module over R.

If $\dim(u^\tau R_1 + v^\tau R_1 + w^\tau R_1) < 9$, then there exist $a, b, c \in R_1 \setminus \{0\}$ such that $u^\tau a + v^\tau b + w^\tau c = 0$. Since τ is defined on R_1 and S, it follows that $ua^{\tau^{-1}} + vb^{\tau^{-1}} + wc^{\tau^{-1}} \in k\Omega$. Since $R/uR + vR + wR$ is a right point module over R corresponding to p, we have $\Omega \notin uR + vR + wR$. Hence, $ua^{\tau^{-1}} + vb^{\tau^{-1}} + wc^{\tau^{-1}} = 0$. Since a, b, c are all nonzero, this implies $\dim(uR_1 + vR_1 + wR_1) < 9$, which is false. Hence $\dim(u^\tau R_1 + v^\tau R_1 + w^\tau R_1) = 9$. It follows that $R/u^\tau R + v^\tau R + w^\tau R$ has a quotient which is a truncated point module of length 3 corresponding to $\tau^{-1}(p)$, so, by Theorem 1.10 and Lemma 1.12, we have $R/u^\tau R + v^\tau R + w^\tau R$ is a right point module over R. ■

2. The Regular Algebras R whose Point Scheme is not the Quadric Q

In this section we classify the algebras R whose point scheme P is not the quadric Q; by Lemma 1.14, such algebras are determined by their geometric data. We show in §2.1 that if $P \neq Q$, then R may be twisted by an automorphism to an algebra R' which maps onto the (commutative) homogeneous coordinate ring $S(Q)$ of Q. In §2.2 the algebras R' are classified (Proposition 2.10), and so are the possible twists of R', which yields the classification of the algebras R in the case that $P \neq Q$ (Theorem 2.14).

2.1. **Normal Elements in R of Degree One.**

We will show that R_1 contains a normal element but the proof requires a few technical results.

Lemma 2.1. *If the point scheme P is not $\mathbb{P}^3$ and if $p \in P_{red} \setminus Q$, then $\tau(p) \neq \sigma|_{P_{red}}(p)$.*

Proof. By hypothesis, R is not a twist of $S(\mathbb{P}^3)$, so there exist $u, v \in R_1$ such that $u^\tau v - v^\tau u = \Omega$. Let $\tilde{\Omega}$ denote a preimage of Ω in $V \otimes V$ such that $\tilde{\Omega} \notin k(u^\tau \otimes v - v^\tau \otimes u)$ but $u^\tau \otimes v - v^\tau \otimes u - \tilde{\Omega} \in W$. Since $W(\Gamma_{\text{red}}) = 0$, we have that $(u^\tau \otimes v - v^\tau \otimes u - \tilde{\Omega})(p, \sigma|_{P_{\text{red}}}(p)) = 0$, where $p \in P_{\text{red}} \setminus Q$. Thus, if $\tau(p) = \sigma|_{P_{\text{red}}}(p)$, then $\tilde{\Omega}(p, \tau(p)) = 0$, which is false since the point scheme of S is Q. ■

Lemma 2.2. *Suppose $P \neq Q$ but $P_{red} = Q$. If $p \in Q$ corresponds to a multiple point of P, then $u^\tau v = v^\tau u$ in R for all $u, v \in R_1$ such that $p \in \mathcal{V}(u, v)$.*

Proof. Suppose $p \in Q$ corresponds to a multiple point of P. By definition of P, we have that p has multiplicity at least two on P. To reinterpret this in terms of the graph of σ, consider $\sigma(p)$ which is computed on the open affine subsets $P \setminus \mathcal{V}(x_i)$ of P by $\sigma(p) = \sigma(x)|_p$ where x and $\sigma(x) \in (S(P)[x_i^{-1}])^4$ where $S(P)$ is the homogeneous coordinate ring of P (see [4]).

Let $u, v \in R_1$ be such that $u(p) = 0 = v(p)$. Since $p \in Q$, we have that $\tau^{-1}(p)$ is the image of $\sigma^{-1}(p)$ in $S(Q)[x_i^{-1}]$, so

$$\sigma^{-1}(p) = \sigma^{-1}(x_1, \ldots, x_4)|_p = (x_1^{\tau^{-1}} + a_1 q, \ldots, x_4^{\tau^{-1}} + a_4 q)|_p$$

where $\mathcal{V}(q) = Q$ and $a_j \in S(P)[x_i^{-1}]$ for $j = 1, \ldots, 4$. It follows that

$$(u^\tau \otimes v - v^\tau \otimes u)(\sigma^{-1}(p), p) = (uv - vu + q(bv - cu))|_p$$

for some $b, c \in S(P)[x_i^{-1}]$ depending on u, v. Thus, $u^\tau \otimes v - v^\tau \otimes u$ vanishes at $(\sigma^{-1}(p), p)$ with at least multiplicity two. However, in R, $u^\tau v - v^\tau u \in k\Omega$, so the result follows, since the point scheme of S is Q. ■

Lemma 2.3. *Let $p \in Q$. There are distinct lines ℓ_1, ℓ_2 in $\mathbb{P}^3$ such that $p \notin \ell_i \not\subset Q$, both lines ℓ_i correspond to line modules over R and the planes determined by p and ℓ_i, for $i = 1, 2$, are distinct.*

Proof. Choose $w \in R_1$ such that $p \notin \mathcal{V}(w)$ and fix $p_1 \in \mathcal{V}(w) \setminus Q$. Pick $u_1, u_2 \in R_1$ such that $p_1 = \mathcal{V}(w, u_1, u_2)$. Then, for $i = 1, 2$, there exists $\alpha_i \in k$ such that $u_i^\tau w - w^\tau u_i = \alpha_i \Omega$. If $\alpha_i = 0$ for $i = 1$ or 2, then for this value of i, set $u' = u_i$. If $\alpha_1\alpha_2 \neq 0$, then set $u' = \alpha_2 u_1 - \alpha_1 u_2$. In either case, $(u')^\tau w = w^\tau u'$ so the line $\ell_1 = \mathcal{V}(w, u')$ corresponds to a line module, and $p \notin \ell_1 \not\subset Q$.

Let X denote the plane determined by p and ℓ_1. Choose $p_2 \in \mathbb{P}^3 \setminus (X \cup Q)$. Let Y be any plane containing p_2 but not containing p. Repeating the above argument for p_2 and Y yields a line $\ell_2 \subset Y$ such that $p_2 \in \ell_2$, $p \notin \ell_2 \not\subset Q$ and ℓ_2 corresponds to a line module. ■

Proposition 2.4. *If the point scheme P is not the quadric Q, then there exists a normal element in R_1.*

Proof. If $P = \mathbb{P}^3$, then R is a twist of $S(\mathbb{P}^3)$, so $x^\tau y = y^\tau x$ for all $x, y \in R_1$. Thus, if $v \in R_1$ is an eigenvector of τ, then v is normal in R. Henceforth, assume $P \neq \mathbb{P}^3$ and $P \neq Q$.

Firstly, assume $P_{\text{red}} \neq Q$ and fix $p \in P_{\text{red}} \setminus Q$. Let $u, v \in R_1$ be linearly independent with $\sigma(p), \tau(p), \tau\sigma\tau^{-1}(p) \in \mathcal{V}(u)$ and $\sigma(p), \tau(p), \tau\sigma\tau^{-2}\sigma(p) \in \mathcal{V}(v)$, where $\sigma(p)$ denotes $\sigma|_{P_{\text{red}}}(p)$. (By Lemma 2.1, $\tau\sigma\tau^{-1}(p) \neq \tau\sigma\tau^{-2}\sigma(p)$, so this is possible.) Since $u^\tau \otimes x - x^\tau \otimes u$ and $v^\tau \otimes x - x^\tau \otimes v$ vanish on $\Gamma_\tau(Q)$ and $(p, \sigma(p))$ for all $x \in R_1$, it follows that

$$W_p = \operatorname{span}\{u^\tau \otimes x - x^\tau \otimes u,\ v^\tau \otimes x - x^\tau \otimes v \ :\ x \in R_1\}$$

is a five-dimensional subspace of W. Moreover, $(u^\tau \otimes x - x^\tau \otimes u)^\tau$ (respectively, $(v^\tau \otimes x - x^\tau \otimes v)^\tau$) vanishes on $\Gamma_\tau(Q)$ and $(\tau^{-1}(p), \sigma\tau^{-1}(p))$ by choice of u (respectively, $\Gamma_\tau(Q)$ and $(\tau^{-2}\sigma(p), \sigma\tau^{-2}\sigma(p))$ by choice of v) for all $x \in R_1$. Hence, $(W_p)^\tau \subset W$ and, since $P \neq \mathbb{P}^3$, we have $(W_p)^\tau = W_p$. In particular, $(v^\tau \otimes x - x^\tau \otimes v)^{\tau^i} \in W$ for all $x \in R_1$, for all $i \in \mathbb{Z}$. Let $z \in R_1$ be an eigenvector of τ. If z, v, v^τ, v^{τ^2} are linearly independent, or if $z \in kv + kv^\tau + kv^{\tau^2}$, then z is normal in R. Otherwise, either $v^{\tau^2} \in kv + kv^\tau$ or $v^\tau \in kv$; in the first case, $kv + kv^\tau$ contains an eigenvector of τ which is normal in R; in the second, v is normal in R.

Now suppose $P_{\text{red}} = Q$ and let $p \in Q$ correspond to a multiple point of P. By Lemma 2.3, there exist two distinct lines ℓ_i which correspond to line modules $M(\ell_i)$ over R such that $p \notin \ell_i \not\subset Q$, and the planes $\mathcal{V}(u), \mathcal{V}(v)$ (where $u, v \in R_1$) determined by (p, ℓ_1) and (p, ℓ_2) respectively are distinct. Thus, there exists $w \in R_1$ such that $p = \mathcal{V}(u, v, w)$ and by Lemma 2.2 we have $u^\tau v = v^\tau u$, $u^\tau w = w^\tau u$ and $w^\tau v = v^\tau w$. Since $\ell_i \not\subset Q$, we have $\Omega \notin \operatorname{Ann} M(\ell_i)$, so that writing $\ell_1 = \mathcal{V}(u, u_1)$ and $\ell_2 = \mathcal{V}(v, v_1)$ for some $u_1, v_1 \in R_1$, we have $u^\tau u_1 = u_1^\tau u$ and $v^\tau v_1 = v_1^\tau v$. Moreover, since $p \notin \ell_i$, the sets $\{u, v, w, u_1\}$ and $\{u, v, w, u_2\}$ are each linearly independent.

It follows that the space

$$W_p = \operatorname{span}\{u^\tau \otimes x - x^\tau \otimes u,\ v^\tau \otimes x - x^\tau \otimes v \ :\ x \in R_1\}$$

is a five-dimensional subspace of W. By Lemma 2.2, $(v^\tau \otimes u - u^\tau \otimes v)^\tau$, $(v^\tau \otimes w - w^\tau \otimes v)^\tau$ and $(u^\tau \otimes w - w^\tau \otimes u)^\tau$ belong to W. We may assume $(v^\tau \otimes x - x^\tau \otimes v)^\tau \in W$ for all $x \in R_1$. (If not, then there exists $y \in R_1$ with $y(p) \neq 0$ and $(w^\tau y - y^\tau w)^\tau = \alpha\Omega$ and $(v^\tau y - y^\tau v)^\tau = \beta\Omega$ where $\alpha, \beta \in k$, $\beta \neq 0$ and we may replace v by $v' = \alpha v - \beta w$.) Similarly, we may assume $(u^\tau \otimes x - x^\tau \otimes u)^\tau \in W$ for all $x \in R_1$. Thus $(W_p)^\tau \subset W$ and, since $P \neq \mathbb{P}^3$, we have $(W_p)^\tau = W_p$. Hence we may continue as above for the case $P_{\text{red}} \neq Q$, using a τ-eigenvector $z \in R_1$ and comparing it with v, v^τ, v^{τ^2} to conclude the result. ■

The normal element of Proposition 2.4 may be viewed as determining τ as the following lemma demonstrates.

Lemma 2.5. *If $\omega \in R_1$ is normal and $\phi \in \mathit{Aut}(R)$ is defined by $\omega x = x^\phi \omega$ for all $x \in R$, then $\phi \in k^\times \tau$.*

Proof. Since ω is normal in R, it is also normal in S. In particular, $\omega s = s^\phi \omega$ for all s in S_1. It follows that $(\omega \otimes s - s^\phi \otimes \omega)(p, \tau(p)) = 0$ for all $p \in Q$. Let $p \in Q \cap \mathcal{V}(\omega)$ and choose $s \in S_1$ such that $s^\phi(p) \neq 0$. Then $\omega^\tau(p) = 0$; that is, τ fixes $Q \cap \mathcal{V}(\omega)$. Since τ is linear, it fixes $\mathcal{V}(\omega)$, so there exists $\beta \in k^\times$ such that $\omega^\tau = \beta\omega$. However in S we have $\omega^\tau s = s^\tau \omega$ for all $s \in S_1$. Thus $s^\phi \omega = \omega s = \beta^{-1} s^\tau \omega$ and so $s^\phi = \beta^{-1} s^\tau$ since S is a domain. ■

Theorem 2.6. *If the point scheme P of R is not the quadric Q, then $\tau \in \mathit{Aut}(R)$. In this case, R is a twist by τ of a regular algebra R' which maps onto the (commutative) homogeneous coordinate ring $S(Q)$ of Q, such that $\mathit{gldim}(R') = 4$ and the Hilbert series of R' is $H_{R'}(t) = (1-t)^{-4}$. Moreover R' contains two linearly independent central elements of degree one.*

Proof. If $P \neq Q$, then R contains a normal element by Proposition 2.4 so Lemma 2.5 implies $\tau \in \operatorname{Aut}(R)$. It follows that the twist R' of R by τ^{-1} is defined, and the automorphism τ' of Q corresponding to R' is the identity map. Hence R' maps onto $S(Q)$; by [23], R' is regular of global dimension four with $H_{R'}(t) = (1-t)^{-4}$.

By the proof of Proposition 2.4, there exist linearly independent elements $v_1, v_2 \in R_1$ such that W contains a five-dimensional subspace generated by $\{v_i^\tau \otimes x - x^\tau \otimes v_i \ :\ x \in R_1, i = 1, 2\}$. It follows that the space W' of defining relations of R' contains a five-dimensional subspace generated by $\{v_i \otimes x - x \otimes v_i \ :\ x \in R'_1, i = 1, 2\}$ which completes the proof. ■

It follows from Theorem 2.6 and [23] that for each $P \neq Q$, if R and R'' are regular algebras corresponding to P which are twists by automorphisms of the same regular algebra R' which maps onto $S(Q)$, then R and R'' are twists by twisting systems of eachother.

This situation arises in [19], for algebras whose point scheme is $Q \cup L$ where L is a line in $\mathbb{P}^3$ that intersects Q at two distinct points. The automorphism τ of Q either preserves the two rulings on Q or interchanges them. In [19] it is shown that algebraic properties of the algebras therein are related to this geometric property, yet both types of algebra may be twisted by automorphisms to one mapping onto $S(Q)$, and so an algebra whose automorphism preserves the rulings on Q may be twisted by a twisting system to one whose corresponding automorphism interchanges the rulings.

Proposition 2.7. *If the point scheme P of R is not the quadric Q, then either*

(a) $P = \mathbb{P}^3$, *or*

(b) $P = Q \cup L$ *where L is a line in $\mathbb{P}^3$ that meets Q in two points (counted with multiplicity), or*

(c) *the point variety P_{red} is the quadric Q and P contains a double line L of (all) multiple points on P, where L corresponds to a line on Q (we denote this situation by $P = Q \uplus L$).*

In cases (b) and (c) there is a regular normalizing sequence $\{v_1, v_2\} \subset R_1$ such that $L = \mathcal{V}(v_1, v_2)$.

Proof. Suppose $P \neq \mathbb{P}^3$ (so R is not a twist of $S(\mathbb{P}^3)$) and that $P \neq Q$. By Theorem 2.6, we may twist R by τ^{-1}, so since P is invariant under twisting, we may assume R has point scheme (P, σ) where τ is the identity on Q. By Theorem 2.6, there exist linearly independent central elements $v_1, v_2 \in R_1$. Let $L = \mathcal{V}(v_1, v_2)$.

Six of the fifteen 4×4 minors of each of the two matrices determined by the defining relations of R have the form $v_i^2 g_{ij}$ where g_{ij} is a polynomial of degree 2 or is identically zero for $j = 1, 2, 3$. If, for $i = 1$ or 2, $g_{ij} \equiv 0$ for all j, then it is straightforward to show that this would imply that the algebra R/Rv_i has point scheme $\mathcal{V}(v_i) \cong \mathbb{P}^2$. However, σ restricts to the point scheme of the algebra R/Rv_i, so in this case, $\sigma \in \mathrm{Aut}(\mathcal{V}(v_i))$ and hence σ is linear on $\mathcal{V}(v_i)$. It would then follow that $\sigma|_{\mathcal{V}(v_i)} =$ identity, which contradicts Lemma 2.1. Hence, for each i, there exists j such that $g_{ij} \neq 0$, and since $Q \subset P$, this g_{ij} vanishes on Q. It follows that $P_{\text{red}} \setminus Q \subset L$ and any multiple points of P lie on L.

On the other hand, the algebra R/Rv_i has three generators and three quadratic defining relations, so its point scheme is a cubic divisor D_i in $\mathbb{P}^2$, whose reduced variety is $P_{\text{red}} \cap \mathcal{V}(v_i)$. It follows that D_i is either the union of a conic on Q with a line L_i not contained in Q, or D_i is the union of two lines on Q where one of the lines is a double line. In the first case, $L_i = L$ since $P_{\text{red}} \setminus Q \subset L$, and the only multiple points are $Q \cap L$, which yields (b). In the second case, since all the multiple points of P lie on L, it follows that the double line is L, so (c) is proved. The last statement follows from Proposition 2.4 and Lemma 1.16. ■

Example 2.8. Examples of algebras of type (b) are the coordinate ring $\mathcal{O}_q(M_2)$ of quantum 2×2 matrices, the coordinate ring $\mathcal{O}_q(\mathfrak{sp}k^4)$ of quantum symplectic 4-dimensional space, and some quantum groups on four generators given in [3], [17, §4b] and [18, §2]. In the case of $\mathcal{O}_q(M_2)$, identifying $\mathbb{P}^3$ with $\mathbb{P}(M_2)$, the quadric Q corresponds to the singular matrices. These examples have $P = Q \cup L$ where $L \cap Q$ consists of two distinct points (for details see [19, Examples 1.5-1.7] and [20]). A. Giaquinto pointed out to us a deformation of $\mathcal{O}(M_2)$, obtained from an "R-matrix" construction, which is the k-algebra $A = k[a, b, c, d]$ with six defining relations

$$\begin{array}{lll} cd - dc = \alpha c^2, & ad - da = \alpha ca + \beta cd, & bd - db = \beta(d^2 + cb - \alpha ca - da), \\ ac - ca = \beta c^2, & bc - cb = \beta dc - \alpha ca, & ab - ba = \alpha(a^2 - ad + cb + \beta cd), \end{array}$$

where $\alpha, \beta \in k$. At generic values of α and β the point scheme of A is $P = Q \cup L$ where $Q = \mathcal{V}(ad - bc)$ and $L = \mathcal{V}(c, a - d)$ which is a tangent line to Q.

In [8], regular central extensions of regular algebras of global dimension three are considered. These are quadratic regular algebras A with a central element $z \in A_1$ such that A/Az is a regular algebra of global dimension three.

Theorem 2.9. *If the point scheme P of R is not the quadric Q, then R is a twist by τ of a regular central extension of a regular algebra of global dimension three.*

Proof. By Theorem 2.6, twisting R by τ^{-1} yields a regular algebra R' with two linearly independent central elements $v_1, v_2 \in R'_1$ and R' maps onto $S(Q)$. If $P = \mathbb{P}^3$, then $R' \cong S(\mathbb{P}^3)$, so the result holds in this case. If $P \neq \mathbb{P}^3$, then, by the proof of Proposition 2.7, $R'/R'v_i$ has point scheme which is the union of a conic C and a line L (possibly, $L \subset C$) and the corresponding automorphism maps C to C and L to L. Hence $R'/R'v_i$ belongs to the classification in [4] (in particular, $R'/R'v_i$ is standard and nondegenerate), so is a regular algebra of global dimension three. ■

2.2. Classification of the Algebras R with Point Scheme $P \neq Q$.

By Theorem 2.6, understanding the special case $\tau =$ identity is a first step towards classifying the algebras R for which $P \neq Q$. In the following result, cases (b) and (b′) correspond to case (b) of Proposition 2.7.

Proposition 2.10. *Suppose the point scheme P of R is not the quadric Q. If $\tau =$ identity, then there exist generators $x_1, \ldots, x_4$ for R such that R has six quadratic defining relations given by one of the following cases. In each case, $Q = \mathcal{V}(x_1x_4 - x_2x_3)$ and $\sigma|_Q = \tau =$ identity.*

(a) *$R = S(\mathbb{P}^3) = k[x_1, \ldots, x_4]$ with defining relations $x_ix_j = x_jx_i$ for $1 \leq i, j \leq 4$. In this case $(P, \sigma) = (\mathbb{P}^3,\ id)$.*

(b) $R = k[x_1, \dots, x_4]$ *with defining relations*

$$\begin{array}{lll} x_4x_3 = x_3x_4, & x_4x_1 = x_1x_4, & x_3x_1 = x_1x_3, \\ x_4x_2 = x_2x_4, & x_3x_2 = \alpha x_2x_3 - (\alpha - 1)x_1x_4, & x_2x_1 = x_1x_2, \end{array}$$

where $\alpha \in k^\times \setminus \{1\}$*. In this case,* $P = Q \cup L$ *where* $L = \mathcal{V}(x_1, x_4)$ *(so* L *intersects* Q *at two distinct points) and* $\sigma|_L(0, x_2, x_3, 0) = (0, \alpha x_2, x_3, 0)$*.*

(b′) $R = k[x_1, \dots, x_4]$ *with defining relations*

$$\begin{array}{lll} x_4x_3 = x_3x_4, & x_4x_1 = x_1x_4, & x_3x_1 - x_1x_3 = \alpha(x_1x_4 - x_2x_3), \\ x_4x_2 = x_2x_4, & x_3x_2 = x_2x_3, & x_2x_1 - x_1x_2 = \alpha(x_1x_4 - x_2x_3), \end{array}$$

where $\alpha \in k^\times$*. In this case,* $P = Q \cup L$ *where* $L = \mathcal{V}(x_2 - x_3, x_4)$ *(so* L *is tangential to* Q*) and* $\sigma|_L(x_1, x_2, x_2, 0) = (x_1 - \alpha x_2, x_2, x_2, 0)$*.*

(c) $R = k[x_1, \dots, x_4]$ *with defining relations*

$$\begin{array}{lll} x_4x_3 = x_3x_4, & x_4x_1 = x_1x_4, & x_3x_1 = x_1x_3, \\ x_4x_2 = x_2x_4, & x_3x_2 = x_2x_3, & x_2x_1 - x_1x_2 = \alpha(x_1x_4 - x_2x_3), \end{array}$$

where $\alpha \in k^\times$*. In this case,* $P = Q \uplus L$ *where* $L = \mathcal{V}(x_3, x_4) \subset Q$ *and* $\sigma \in Aut(P)$ *is uniquely determined by its action on the affine open sets* $P \setminus \mathcal{V}(x_i)$ *as follows: on* $P \setminus \mathcal{V}(x_1)$ *we have* $\sigma(x_1, \dots, x_4) = (x_1, x_2 - \alpha x_1^{-1}(x_1x_4 - x_2x_3), x_3, x_4)$*, and on* $P \setminus \mathcal{V}(x_2)$ *we have* $\sigma(x_1, \dots, x_4) = (x_1 + \alpha x_2^{-1}(x_1x_4 - x_2x_3), x_2, x_3, x_4)$*, but* σ *is the identity on* $P \setminus \mathcal{V}(x_3)$ *and on* $P \setminus \mathcal{V}(x_4)$*.*

Proof. By Proposition 2.7 we have $P = \mathbb{P}^3$ or $P = Q \cup L$ or $P = Q \uplus L$ where L is a line in $\mathbb{P}^3$.

(a) If $P = \mathbb{P}^3$, then $\sigma \in \mathrm{Aut}(\mathbb{P}^3)$ and is determined by $\tau(=$ identity). Thus, $\sigma =$ identity and so $R = S(\mathbb{P}^3)$.

(b) If $P = Q \cup L$ where $Q \cap L$ consists of two distinct points, then $\sigma|_{Q\cap L} = \tau|_{Q\cap L} =$ identity, so we may apply [19, Lemma 1.3(a) and Proposition 2.3] to yield (b).

(b′) Suppose $P = Q \cup L$ where L is tangential to Q. By an argument similar to that of [19, Lemma 1.1], we may assume there exist linearly independent elements $x_1, \dots, x_4 \in V$ with respect to which $Q = \mathcal{V}(x_1x_4 - x_2x_3)$ and $L = \mathcal{V}(x_2 - x_3, x_4)$. By the proof of Proposition 2.7, the elements $x_2 - x_3$ and x_4 are central in R, which determines five of the defining relations. In particular, $x_1x_3 - x_3x_1 = x_1x_2 - x_2x_1 \in k^\times\Omega$ since $P \neq \mathbb{P}^3$. Moreover $x_1x_4 - x_2x_3 \in k\Omega$ since it vanishes on the graph of τ and it is nonzero in R since (by Theorem 2.9) R/Rx_4 is a domain. It follows that there exists $\alpha \in k^\times$ such that $x_2x_1 - x_1x_2 = \alpha(x_1x_4 - x_2x_3)$. To finish (b′) it remains to prove that an algebra with such defining relations is indeed a regular algebra of global dimension four with Hilbert series $H(t) = (1 - t)^{-4}$. By Theorem 2.9, we may apply [8] as follows. Let

$$\begin{aligned} f = (f_1, f_2, f_3) &= (x_3x_2 - x_2x_3,\ x_1x_3 - x_3x_1 - \alpha x_3x_2,\ x_2x_1 - x_1x_2 + \alpha x_2x_3), \\ l = (l_1, l_2, l_3) &= (0,\ \alpha x_1,\ -\alpha x_1), \\ x^* = (x_1^*, x_2^*, x_3^*) &= (x_1 + \alpha x_2 + \alpha x_3,\ x_2,\ x_3). \end{aligned}$$

By [8, Theorem 3.1.3], the given algebra is regular of global dimension four if and only if there exists a solution $(\gamma_1, \gamma_2, \gamma_3)$ to the equations

$$\sum \gamma_i f_i = \sum (x_i l_i - l_i x_i^*) \qquad \text{and} \qquad \sum \gamma_i l_i = 0.$$

The solution is $(\gamma_1, \gamma_2, \gamma_3) = (\alpha^2, \alpha, \alpha)$ so the algebra is regular. It follows from [5] that the algebra is a domain, so its Hilbert series has the required form.

(c) Suppose $P = Q \uplus L$ where $L \subset Q$. By an argument similar to that of [19, Lemma 1.1], we may assume there exist linearly independent elements $x_1, \ldots, x_4 \in V$ with respect to which $Q = \mathcal{V}(x_1x_4 - x_2x_3)$ and $L = \mathcal{V}(x_3, x_4)$. As before in (b′), the elements x_3 and x_4 are central in R, which determines five of the defining relations. An argument similar to (b′) yields $x_2x_1 - x_1x_2 \in k^\times\Omega$ and $x_1x_4 - x_2x_3 \in k^\times\Omega$, so $x_2x_1 - x_1x_2 = \alpha(x_1x_4 - x_2x_3)$ for some $\alpha \in k^\times$. As in (b′), one can show using [8, Theorem 3.1.3] that such relations define a regular algebra (in this case take $f_2 = x_1x_3 - x_3x_1$, $l_2 = 0$, $x_1^* = x_1 + \alpha x_3$ and $(\gamma_1, \gamma_2, \gamma_3) = (0, \alpha, 0)$).

By Lemma 1.14, R is determined by the geometric data (P, σ), so to determine the action of σ on P we consider the induced action on the affine open sets $P \setminus \mathcal{V}(x_i)$ for $i = 1, \ldots, 4$; more precisely, we consider the action of σ on the localizations

$$S(P)[x_i^{-1}] = \left(\frac{S(\mathbb{P}^3)}{\langle x_3(x_1x_4 - x_2x_3), x_4(x_1x_4 - x_2x_3)\rangle} \right) [x_i^{-1}]$$

of the homogeneous coordinate ring $S(P)$ of P. The following method is given in [4]. Write the six defining relations of R in the form $M(x_1\ x_2\ x_3\ x_4)^T = 0$ where M is the 6×4 matrix

$$M = \begin{bmatrix} 0 & 0 & x_4 & -x_3 \\ 0 & x_4 & 0 & -x_2 \\ x_4 & 0 & 0 & -x_1 \\ 0 & x_3 & -x_2 & 0 \\ x_3 & 0 & -x_1 & 0 \\ x_2 & -x_1 & \alpha x_2 & -\alpha x_1 \end{bmatrix}.$$

Since $\mathcal{V}(W)$ is the graph of σ, it follows that if $p \in P$, then $\sigma(p)$ is orthogonal to every row of the matrix $M|_p$, the matrix obtained by evaluating the entries of M at p. The rank of $M|_p$ is exactly three since $p \in P$ and $\mathcal{V}(W)$ is the graph of an automorphism. The standard method to find a vector orthogonal to the rows of a matrix implies that if $p \in P \setminus \mathcal{V}(x_i)$, then $\sigma(p)$ is given by

$$(\sigma(p))_j = ((-1)^j x_i^{-2} m_j^{r_1r_2r_3})|_p$$

where $m_j^{r_1r_2r_3}$ is the 3×3 minor of M formed by deleting column j and deleting rows r_1, r_2, r_3 which have no x_i entry. This gives a point of $P \setminus \mathcal{V}(x_i)$ since the ith coordinate is $\pm x_i|_p$ which is nonzero. Hence on $S(P)[x_1^{-1}]$ we have $\sigma(x_1, \ldots, x_4) = (x_1, x_2 - \alpha x_1^{-1}(x_1x_4 - x_2x_3), x_3, x_4)$, on $S(P)[x_2^{-1}]$ we have $\sigma(x_1, \ldots, x_4) = (x_1 +$

$\alpha x_2^{-1}(x_1x_4 - x_2x_3), x_2, x_3, x_4)$, but σ is the identity on $S(P)[x_3^{-1}]$ and on $S(P)[x_4^{-1}]$. Notice that σ depends on α, but σ restricted to P_{red} is the identity (as expected). ∎

Remark 2.11. The algebras in (b′) are all isomorphic to eachother and so we may take $\alpha = 1$; the same is true for (c). However, this would obscure the importance of σ and so we retain "α" in the rest of this section.

The next Lemma is analogous to [5, Proposition 8.8] and will be used to classify all twists of the above algebras.

Lemma 2.12. *Let $\mathcal{P}$ be a subscheme of $\mathbb{P}^n$ and $\theta \in Aut(\mathcal{P})$ and let $A = T(U)/\langle W\rangle$ be the quadratic algebra determined by the geometric data $(\mathcal{P}, \theta)$. Suppose ϕ is a graded degree zero automorphism of $T(U)$. If $\phi \in Aut(\mathcal{P})$ and $\phi \circ \theta = \theta \circ \phi$ on $\mathcal{P}$, then $\phi \in Aut(A)$. If $\mathcal{V}(W)$ is the graph of θ, then the converse holds.*

Proof. The result follows from the proof of [5, Proposition 8.8]. ∎

Corollary 2.13. *We have $\tau \in Aut(P)$ and τ commutes with σ on P.*

Proof. If $P = Q$, then $\tau = \sigma$, so there is nothing to prove. If $P \neq Q$, then R is determined by the geometric data (P, σ) by Lemma 1.14 and $\tau \in \mathrm{Aut}(R)$ by Theorem 2.6 so Lemma 2.12 applies. ∎

If a quadratic algebra A is determined by geometric data $(\mathcal{P}, \theta)$, then the twist A^ϕ of A by a graded degree zero automorphism ϕ (recall Definition 1.2) is determined by the geometric data $(\mathcal{P}, \phi \circ \theta)$.

By Proposition 2.7 and Theorem 2.6, to classify the algebras R whose point scheme P is not Q, it remains to find all the graded degree zero automorphisms of the algebras in Proposition 2.10 and to twist those algebras by such an automorphism.

For the following result, recall that x^τ denotes $\tau(x) = x \circ \tau$ for all $x \in R_1$.

Theorem 2.14. *The regular algebra R is isomorphic to one of the following algebras if and only if the point scheme P of R is not the quadric Q.*

(a) *$R = k[x_1, \dots, x_4]$ with defining relations $x_i^\tau x_j = x_j^\tau x_i$ for $1 \le i, j \le 4$ for all $\tau \in Aut(\mathbb{P}^3)$. In this case, R is the twist of the polynomial ring $S(\mathbb{P}^3)$ by τ, so $(P, \sigma) = (\mathbb{P}^3, \tau)$.*

(b) *$R = k[x_1, \dots, x_4]$ with defining relations*

$$\begin{array}{lll} x_4^\tau x_3 = x_3^\tau x_4, & x_4^\tau x_1 = x_1^\tau x_4, & x_3^\tau x_1 = x_1^\tau x_3, \\ x_4^\tau x_2 = x_2^\tau x_4, & x_3^\tau x_2 = \alpha x_2^\tau x_3 - (\alpha - 1) x_1^\tau x_4, & x_2^\tau x_1 = x_1^\tau x_2, \end{array}$$

for all $\alpha \in k^\times \setminus \{1\}$ *where* τ *is given (with respect to a basis dual to the* x_i*) by*

$$\tau \in k^\times \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & ab \end{pmatrix} \qquad or \qquad \tau \in k^\times \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ ab & 0 & 0 & 0 \end{pmatrix} \tag{1}$$

for all $a, b \in k^\times$*; if* $\alpha = -1$*, then* τ *may be given by* $(*)$ *or*

$$\tau \in k^\times \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & 0 & ab \end{pmatrix} \qquad or \qquad \tau \in k^\times \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & a & 0 \\ 0 & b & 0 & 0 \\ ab & 0 & 0 & 0 \end{pmatrix}$$

for all $a, b \in k^\times$*. In this case,* $P = Q \cup L$ *where* $L = \mathcal{V}(x_1, x_4)$ *(so* L *intersects* Q *at two distinct points),* $\sigma|_Q = \tau$ *and* $\sigma|_L(0, x_2, x_3, 0) = \tau(0, \alpha x_2, x_3, 0)$*.*

(b′) $R = k[x_1, \dots, x_4]$ *with defining relations*

$$\begin{array}{lll} x_4^\tau x_3 = x_3^\tau x_4, & x_4^\tau x_1 = x_1^\tau x_4, & x_3^\tau x_1 - x_1^\tau x_3 = \alpha(x_1^\tau x_4 - x_2^\tau x_3), \\ x_4^\tau x_2 = x_2^\tau x_4, & x_3^\tau x_2 = x_2^\tau x_3, & x_2^\tau x_1 - x_1^\tau x_2 = \alpha(x_1^\tau x_4 - x_2^\tau x_3), \end{array}$$

for all $\alpha \in k^\times$ *where* τ *is given (with respect to a basis dual to the* x_i*) by*

$$\tau \in k^\times \begin{pmatrix} 1 & a & b & ab \\ 0 & 0 & 1 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad or \qquad \tau \in k^\times \begin{pmatrix} 1 & a & b & ab \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

for all $a, b \in k$*. In this case,* $P = Q \cup L$ *where* $L = \mathcal{V}(x_2 - x_3, x_4)$ *(so* L *is tangential to* Q*),* $\sigma|_Q = \tau$ *and* $\sigma|_L(x_1, x_2, x_2, 0) = \tau(x_1 - \alpha x_2, x_2, x_2, 0)$*.*

(c) $R = k[x_1, \dots, x_4]$ *with defining relations*

$$\begin{array}{lll} x_4^\tau x_3 = x_3^\tau x_4, & x_4^\tau x_1 = x_1^\tau x_4, & x_3^\tau x_1 = x_1^\tau x_3, \\ x_4^\tau x_2 = x_2^\tau x_4, & x_3^\tau x_2 = x_2^\tau x_3, & x_2^\tau x_1 - x_1^\tau x_2 = \alpha(x_1^\tau x_4 - x_2^\tau x_3), \end{array}$$

for all $\alpha \in k^\times$ *where* τ *is given (with respect to a basis dual to the* x_i*) by*

$$\tau \in k^\times \begin{pmatrix} a & b & \lambda a & \lambda b \\ c & d & \lambda c & \lambda d \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{pmatrix}$$

for all $a, b, c, d, \lambda \in k$ *such that* $ad - bc \neq 0$*. In this case,* $P = Q \uplus L$ *where* $L = \mathcal{V}(x_3, x_4) \subset Q$ *and* $\sigma \in Aut(P)$ *is uniquely determined by its action on the affine open sets* $P \setminus \mathcal{V}(x_i)$ *as follows: on* $P \setminus \mathcal{V}(x_1)$ *we have* $\sigma(x_1, \dots, x_4) = \tau(x_1, x_2 - \alpha x_1^{-1}(x_1 x_4 - x_2 x_3), x_3, x_4)$*, and on* $P \setminus \mathcal{V}(x_2)$ *we have* $\sigma(x_1, \dots, x_4) = \tau(x_1 + \alpha x_2^{-1}(x_1 x_4 - x_2 x_3), x_2, x_3, x_4)$*, but* $\sigma = \tau$ *on* $P \setminus \mathcal{V}(x_3)$ *and on* $P \setminus \mathcal{V}(x_4)$*.*

Proof. By Theorem 2.6 and Proposition 2.10, we have $P \neq Q$ if and only if R is a twist by τ of one of the algebras given in Proposition 2.10. Thus it suffices to classify the graded degree zero automorphisms of the algebras listed in Proposition 2.10 which are determined by (P, σ) where $\sigma|_Q =$ identity. Lemma 2.12 applies to (a), (b) and (b′), so that the matrix for τ is determined by the conditions: $\tau \in \mathrm{Aut}(Q)$, $\tau \in \mathrm{Aut}(L)$ and $\tau|_L \circ \sigma|_L = \sigma|_L \circ \tau|_L$ (since $\sigma|_Q =$ identity). A computation completes the proof for (a), (b) and (b′).

For (c), Lemma 2.12 implies that $\tau \in \mathrm{Aut}(Q)$ and $\tau \in \mathrm{Aut}(L)$ but it is harder to work with the commutativity of τ and σ. However a generic element of $\mathrm{Aut}(Q) \cap \mathrm{Aut}(L)$ preserves the rulings on Q and so, with respect to a basis dual to $\{x_1, \ldots, x_4\}$ of Proposition 2.10, such an element may be given by the matrix

$$\begin{pmatrix} a\lambda_1 & b\lambda_1 & a\lambda_2 & b\lambda_2 \\ c\lambda_1 & d\lambda_1 & c\lambda_2 & d\lambda_2 \\ 0 & 0 & a\lambda_3 & b\lambda_3 \\ 0 & 0 & c\lambda_3 & d\lambda_3 \end{pmatrix}$$

where $a, b, c, d, \lambda_i \in k$, $ad - bc \neq 0$ and $\lambda_1\lambda_3 \neq 0$. Applying this matrix to the defining relations yields the result. ■

In case (b), the algebras obtained from twists by the matrices ($*$) were the object of study in [19]; the algebras obtained by twisting by the other two types of matrices (where $\alpha = -1$) did not arise in [19] since, in contrast to the situation there, $\tau|_{Q\cap L}$ is not assumed to be the identity.

Recall that the isomorphism classes of the left (respectively, right) point modules over R are in one-to-one correspondence with P_{red} (Lemma 1.12). The following result concerning the line modules generalizes the analogous result for the examples in [19, 21]; in particular, if $P = Q \cup L$, then L corresponds to a line (bi)module.

Proposition 2.15. *If $P = \mathbb{P}^3$, then the isomorphism classes of left (respectively, right) line modules over R are in one-to-one correspondence with the lines in $\mathbb{P}^3$. If $P = Q \cup L$ or if $P = Q \uplus L$, then the isomorphism classes of left (respectively, right) line modules over R are in one-to-one correspondence with the lines in $\mathbb{P}^3$ that either lie on Q or intersect L.*

Proof. If R' is a twist of R then $\mathrm{Proj}\, R \cong \mathrm{Proj}\, R'$, so there is a one-to-one correspondence between the isomorphism classes of left (respectively, right) line modules over R and those over R'. Thus, by Theorem 2.6 we may assume $\tau =$ identity. By Proposition 2.10, if $P = \mathbb{P}^3$, then $R = S(\mathbb{P}^3)$ and so the first statement holds. If $P = Q \cup L$ or if $P = Q \uplus L$, then the result may be proved either by using [8, Theorem 5.1.6] (see for example [21, Theorem 4.7]), or by using purely geometric techniques as follows.

The lines on Q correspond to line modules since they do over $S(Q)$, so it remains to classify the lines outside Q which correspond to line modules. By [11] and Corollary 1.6, a line $\ell = \mathcal{V}(x, y) \subset \mathbb{P}^3$, where $x, y \in R_1$, corresponds to a left line module $M(\ell)$ if and only if $R_1x \cap R_1y \neq 0$, in which case $M(\ell) \cong R/Rx + Ry$. If $\ell \not\subset Q$ is a line in $\mathbb{P}^3$ which intersects L, then there exists $x \in R_1$ such that $x(\ell) = 0 = x(L)$. Since $x(L) = 0$, x is central and so $R_1x \cap R_1y \neq 0$ for all $y \in R_1$, so that ℓ corresponds to a left line module. Conversely suppose $\ell = \mathcal{V}(x, y) \not\subset Q$ is a line in $\mathbb{P}^3$ which corresponds to a left line module.

If $xy - yx \in k^\times \Omega$, then $\Omega \in Rx + Ry$, so ℓ corresponds to a line module over $S(Q)$ which is false. Hence $xy = yx$. A computation implies that we may assume $x \in kv_1 \oplus kv_2$ where v_1, v_2 are the linearly independent central elements of Theorem 2.6 such that $L = \mathcal{V}(v_1, v_2)$. It follows that L is contained in the plane $\mathcal{V}(x)$ which also contains ℓ, so ℓ intersects L. ■

A consequence of this result is that a line which corresponds to a left (respectively, right) line module also corresponds to a right (respectively, left) line module. This contrasts with the case $P = Q$ which is discussed at the end of §4.

3. The Koszul Duals of S and R

In §4 we show that the techniques of §2 are not successful in classifying the algebras R if $P = Q$. Instead we use the Koszul duals of S and R as defined in Definition 1.3. The purpose of this section is to find potential defining relations of R which depend on certain parameters, and to find necessary conditions on the parameters. An important tool is the existence of a normal, regular element $\omega \in S^!_2$ which plays a role dual to Ω. Throughout this section, the symbol R is reserved for a regular algebra satisfying the hypotheses of §1.1. This section applies to all such R with $P \neq \mathbb{P}^3$.

Lemma 3.1. *If $P \neq \mathbb{P}^3$, then there exists a choice of generators $x_1, \dots, x_4$ for R such that the defining relations of R may be written:*

$$\begin{aligned} x_1^\tau x_4 &= x_2^\tau x_3, & x_1^\tau x_4 - x_4^\tau x_1 &= \alpha(x_1^\tau x_2 - x_2^\tau x_1),\\ x_1^\tau x_3 &= x_3^\tau x_1, & x_2^\tau x_3 - x_3^\tau x_2 &= \beta(x_1^\tau x_2 - x_2^\tau x_1),\\ x_3^\tau x_4 - x_4^\tau x_3 &= x_1^\tau x_2 - x_2^\tau x_1, & x_2^\tau x_4 - x_4^\tau x_2 &= \gamma(x_1^\tau x_2 - x_2^\tau x_1), \end{aligned}$$

where $\alpha, \beta \in k^\times$, $\gamma \in k$. Moreover, we may assume $Q = \mathcal{V}(x_1x_4 - x_2x_3)$ and $\Omega = x_2^\tau x_1 - x_1^\tau x_2$.

Proof. Suppose $P \neq \mathbb{P}^3$. There exist linearly independent elements $x_1, x_2 \in R_1$ such that $\mathcal{V}(x_1, x_2) \subset Q$ but $x_1^\tau x_2 - x_2^\tau x_1 \neq 0$. Then $\mathcal{V}(x_1, x_2)^{\tau^{-1}}$ corresponds to a line module, so by [11, Proposition 2.8] and Corollary 1.6, there exist $x_3, x_4 \in R_1$ such that $x_1^\tau x_4 = x_2^\tau x_3$. By choice of x_1, x_2, we have $x_4 \notin kx_2$ and $x_3 \notin kx_1$. It follows that $x_1^\tau x_4 - x_2^\tau x_3$ vanishes on $\Gamma_\tau(Q)$, so the commutative polynomial $x_1x_4 - x_2x_3$ vanishes on Q. Since Q is nonsingular and irreducible, we have $x_1, \dots, x_4$ are linearly independent and $Q = \mathcal{V}(x_1x_4 - x_2x_3)$. Thus $\mathcal{V}(x_3, x_4) \subset Q$, so by [11] there exist unique elements $a, b \in R_1$ (up to scalar multiples) such that $ax_4 = bx_3$ in R. It follows that $a \in kx_1^\tau$ and $b \in kx_2^\tau$ so that $x_3^\tau x_4 - x_4^\tau x_3 \neq 0$ and so is a multiple of $x_1^\tau x_2 - x_2^\tau x_1$ (since both are a multiple of Ω). By rescaling $x_1, \dots, x_4$ if necessary we may assume: $Q = \mathcal{V}(x_1x_4 - x_2x_3)$, and (in R) $x_1^\tau x_4 = x_2^\tau x_3$ and $x_3^\tau x_4 - x_4^\tau x_3 = x_1^\tau x_2 - x_2^\tau x_1$.

For all $\delta \in k$, we have $\mathcal{V}(x_1 + \delta x_2, x_3 + \delta x_4) \subset Q$ and $(x_1 + \delta x_2)^\tau (x_3 + \delta x_4) - (x_3 + \delta x_4)^\tau (x_1 + \delta x_2) = \epsilon\Omega$ for some $\epsilon \in k$, where ϵ depends on δ. Either $\epsilon \equiv 0$ or ϵ is a quadratic polynomial in δ, so there exists at least one value of δ such that $\epsilon = 0$. For such a value of δ we may replace $x_1 + \delta x_2$ by a new x_1 and $x_3 + \delta x_4$ by a new x_3, to yield $x_1^\tau x_3 = x_3^\tau x_1$. Moreover with this new choice of x_1, x_3 we still have $Q = \mathcal{V}(x_1x_4 - x_2x_3)$ and (in R) $x_1^\tau x_4 = x_2^\tau x_3$ and $x_3^\tau x_4 - x_4^\tau x_3 = x_1^\tau x_2 - x_2^\tau x_1$. It follows from [11] that $x_1^\tau x_4 - x_3^\tau x_2 \neq 0$ and $x_2^\tau x_3 - x_4^\tau x_1 \neq 0$ which gives $x_2^\tau x_3 - x_3^\tau x_2 \neq 0$ and $x_1^\tau x_4 - x_4^\tau x_1 \neq 0$. The result follows. ■

Recall Definitions 1.3 and 1.4.

Lemma 3.2. *We have $R^! \cong S^!/\langle\omega\rangle$ where $\omega \in S^!_2$, $\omega \in W^\perp$ and $\omega \cdot \Omega = 1$. The element ω is normal and 1-regular in $S^!$.*

Proof. The first statement holds by definition of $R^!$ and $S^!$. The second is [16, Lemma 2.5] since Ω is normal and regular in R. ■

If $\psi \in \operatorname{Aut}(\mathbb{P}^3) = \operatorname{Aut}(\mathbb{P}(V^*))$, then ψ may be viewed as an automorphism of both $T(V)$ and $T(V^*)$ in the obvious way. It follows that ψ is an automorphism of R (respectively, S) if and only if ψ is an automorphism of $R^!$ (respectively, $S^!$). This proves part of the next lemma.

Lemma 3.3.
(a) If $\psi \in Aut(S^!)$, then $\psi \circ \tau \in k\tau \circ \psi$.
(b) If $\psi \in Aut(S^!)$ and if $S = A/\langle\Omega\rangle$ where A is any quadratic algebra and $\Omega \in A_2$, then $\psi \in Aut(A)$ if and only if $\omega^\psi \in k^\times\omega$ where $A^! = S^!/\langle\omega\rangle$.

Proof. If $\psi \in \operatorname{Aut}(S^!)$, then $\psi \in \operatorname{Aut}(S)$ and S is determined by the geometric data (Q, τ). By Lemma 2.12, ψ and τ commute on Q and, hence, on $\mathbb{P}^3$. ■

By Lemma 3.2, ω is normal in $S^!$, so it determines a map $\theta \in \operatorname{Aut}(S^!)$ by $\omega a = a^\theta\omega$ for all $a \in S^!$. Since $\theta \in \operatorname{Aut}(S^!)$ and $\omega^\theta = \omega$, we have $\theta \circ \tau \in k\tau \circ \theta$ and $\theta \in \operatorname{Aut}(R)$ by Lemma 3.3. The element ω and the map θ are crucial to the analysis in §4.

We next give an explicit description of ω and θ. Let $x_1, \ldots, x_4$ be the generators of R from Lemma 3.1 so that $Q = \mathcal{V}(x_1x_4 - x_2x_3)$ and let $\eta_1, \ldots, \eta_4$ be a dual basis to $x_1, \ldots, x_4$. Then the defining relations of S are

$$\begin{array}{llll} x_1^\tau x_2 = x_2^\tau x_1, & x_2^\tau x_3 = x_3^\tau x_2, & x_3^\tau x_4 = x_4^\tau x_3, & x_1^\tau x_4 = x_2^\tau x_3, \\ x_1^\tau x_3 = x_3^\tau x_1, & x_2^\tau x_4 = x_4^\tau x_2, & x_1^\tau x_4 = x_4^\tau x_1, & \end{array}$$

and the defining relations of $S^!$ are

$$\begin{array}{llll} \eta_1^{\tau^{-1}}\eta_1 = 0, & \eta_3^{\tau^{-1}}\eta_3 = 0, & \eta_2^{\tau^{-1}}\eta_1 = -\eta_1^{\tau^{-1}}\eta_2, & \eta_2^{\tau^{-1}}\eta_4 = -\eta_4^{\tau^{-1}}\eta_2, \\ \eta_2^{\tau^{-1}}\eta_2 = 0, & \eta_4^{\tau^{-1}}\eta_4 = 0, & \eta_3^{\tau^{-1}}\eta_1 = -\eta_1^{\tau^{-1}}\eta_3, & \eta_3^{\tau^{-1}}\eta_4 = -\eta_4^{\tau^{-1}}\eta_3, \end{array}$$
$$\eta_4^{\tau^{-1}}\eta_1 + \eta_1^{\tau^{-1}}\eta_4 + \eta_2^{\tau^{-1}}\eta_3 + \eta_3^{\tau^{-1}}\eta_2 = 0,$$

and $\tau \in \operatorname{Aut}(S^!)$ since $\tau \in \operatorname{Aut}(S)$. With these defining relations and those in Lemma 3.1, it follows that

$$\omega = \eta_4^{\tau^{-1}}\eta_3 + \eta_2^{\tau^{-1}}\eta_1 + \alpha\eta_4^{\tau^{-1}}\eta_1 + \beta\eta_3^{\tau^{-1}}\eta_2 + \gamma\eta_4^{\tau^{-1}}\eta_2,$$

up to a scalar multiple.

Lemma 3.4. *We have $\theta = \Phi \circ \tau^{-2}$ where (with respect to the basis $\eta_1, \dots, \eta_4$) Φ is determined up to a scalar multiple by the matrix*

$$\begin{bmatrix} 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & \lambda_3\gamma\beta^{-1} & \lambda_4 \\ \lambda_1\beta & 0 & \lambda_3(\alpha-\beta) & 0 \\ \lambda_1\gamma & \lambda_2 & \lambda_3(\alpha-\beta)\gamma\beta^{-1} & \lambda_4(\alpha-\beta) \end{bmatrix}$$

where $\lambda_i \in k^\times$ for $i = 1, \dots, 4$.

Proof. Since $\omega\eta_i = \eta_i^\theta\omega$, it follows that $0 = \omega\eta_i\eta_i^\tau = \eta_i^\theta\omega\eta_i^\tau$ since $\tau \in \mathrm{Aut}(S^!)$. To prove the result, compute $\omega\eta_i^\tau$ for each i, and find which element of degree one multiplies it on the left to give zero. ■

Lemma 3.5. *We have $\alpha = \beta$ or $(\alpha, \beta) = (1, -1)$. Moreover, $\lambda_2 = -\lambda_1\alpha$, $\lambda_3 = -\lambda_1\beta$ and $\lambda_4 = \lambda_1\alpha$.*

Proof. Since $\omega\eta_i = \eta_i^\theta\omega$, we may use θ to rewrite the right hand side. This yields four expressions for ω:

$$\begin{aligned}
\omega &= \lambda_1(\beta\eta_3^{\tau^{-2}}\eta_2^{\tau^{-1}} + \alpha\beta\eta_3^{\tau^{-2}}\eta_4^{\tau^{-1}} - \gamma\eta_2^{\tau^{-2}}\eta_4^{\tau^{-1}} + a\eta_1^{\tau^{-1}}) &(1)\\
\omega &= \lambda_2(-\eta_4^{\tau^{-2}}\eta_1^{\tau^{-1}} - \beta\eta_3^{\tau^{-2}}\eta_4^{\tau^{-1}} + b\eta_2^{\tau^{-1}}) &(2)\\
\omega &= \lambda_3(\eta_1^{\tau^{-2}}\eta_4^{\tau^{-1}} + (\alpha-\beta)\eta_4^{\tau^{-2}}\eta_1^{\tau^{-1}} + \gamma\beta^{-1}\eta_2^{\tau^{-2}}\eta_4^{\tau^{-1}} - \alpha\eta_1^{\tau^{-2}}\eta_2^{\tau^{-1}} + c\eta_3^{\tau^{-1}}) &(3)\\
\omega &= \lambda_4(-\eta_2^{\tau^{-2}}\eta_3^{\tau^{-1}} + \beta(\alpha-\beta)\eta_3^{\tau^{-2}}\eta_2^{\tau^{-1}} + \beta\eta_1^{\tau^{-2}}\eta_2^{\tau^{-1}} + d\eta_4^{\tau^{-1}}) &(4)
\end{aligned}$$

where $a, \dots, d \in S_1^!$. The elements $\eta_1^{\tau^{-2}}\eta_4^{\tau^{-1}}, \eta_4^{\tau^{-2}}\eta_1^{\tau^{-1}}, \eta_3^{\tau^{-2}}\eta_2^{\tau^{-1}}, \eta_2^{\tau^{-2}}\eta_3^{\tau^{-1}}$ are linearly dependent, so we may eliminate $\eta_2^{\tau^{-2}}\eta_3^{\tau^{-1}}$ from the expressions for ω and equate the coefficients of the corresponding terms.

The coefficients of $\eta_3^{\tau^{-2}}\eta_4^{\tau^{-1}}$ and $\eta_1^{\tau^{-2}}\eta_2^{\tau^{-1}}$ imply that $\lambda_2 = -\lambda_1\alpha$ and $\lambda_4 = -\alpha\beta^{-1}\lambda_3$. In (3), the coefficients of $\eta_2^{\tau^{-2}}\eta_3^{\tau^{-1}}$ and $\eta_1^{\tau^{-2}}\eta_4^{\tau^{-1}}$ are equal, and similarly for (4), so the coefficient of $\eta_3^{\tau^{-2}}\eta_2^{\tau^{-1}}$ in (1) and (3) gives $\lambda_3 = -\lambda_1\beta$. Moreover the coefficients of $\eta_3^{\tau^{-2}}\eta_2^{\tau^{-1}}$ in (3) and (4), and of $\eta_4^{\tau^{-2}}\eta_1^{\tau^{-1}}$ in (2) and (3), imply respectively that

$$(\alpha + \beta^{-1})(\alpha - \beta) = 0 \qquad \text{and} \qquad (1 + \beta)(\alpha - \beta) = 0,$$

which completes the proof. ■

Corollary 3.6.
(a) If $\alpha = \beta$, then

$$\omega \in k^\times(\eta_4^{\tau^{-2}}\eta_3^{\tau^{-1}} + \eta_2^{\tau^{-2}}\eta_1^{\tau^{-1}} - \alpha^{-1}\eta_4^{\tau^{-2}}\eta_1^{\tau^{-1}} - \alpha^{-1}\eta_3^{\tau^{-2}}\eta_2^{\tau^{-1}} - \gamma\alpha^{-2}\eta_4^{\tau^{-2}}\eta_2^{\tau^{-1}}),$$

but if $(\alpha, \beta) = (1, -1)$, then

$$\omega \in k^\times(\eta_4^{\tau^{-2}}\eta_3^{\tau^{-1}} + \eta_2^{\tau^{-2}}\eta_1^{\tau^{-1}} + \eta_4^{\tau^{-2}}\eta_1^{\tau^{-1}} - \eta_3^{\tau^{-2}}\eta_2^{\tau^{-1}} + \gamma\eta_4^{\tau^{-2}}\eta_2^{\tau^{-1}}).$$

(b) If $\alpha = \beta = \pm\sqrt{-1}$ or if $(\alpha, \beta) = (1, -1)$, then $\omega^\tau \in k\omega$ and $\tau \in Aut(R)$.

Proof. (a) follows from the proof of Lemma 3.5. The first part of (b) is a computation and yields the second part by Lemma 3.3. ■

We will show in §4 that if $P = Q$, then $\tau \notin \mathrm{Aut}(R)$ but $\tau^2 \in \mathrm{Aut}(R)$.

4. The Regular Algebras R whose Point Scheme is the quadric Q

Throughout this section, we assume $P = Q$, and the symbol R is reserved for a regular algebra satisfying the hypotheses of §1.1. This section is in three parts. In §4.1 we show that, unlike the $P \neq Q$ case, there are no normal elements in R_1 nor is τ an automorphism of R (see Lemma 4.1 and Proposition 4.2). As such, the techniques of §2, which were used to classify R in the $P \neq Q$ case, cannot be applied if $P = Q$. Instead, in §4.2, we use the material from §3 on the Koszul duals of R and S to give defining relations for R which depend on τ, and we find necessary conditions on τ. In §4.3, we show that any algebra which satisfies such relations (with the necessary conditions on τ) twists by a twisting system to one particular algebra. Our main theorem is Theorem 4.13 which gives a classification of the algebras R where $P = Q$.

4.1. **No Normal Elements in R of Degree One.**

Lemma 4.1. *If the point scheme P is the quadric Q, then R has no normal elements in R_1.*

Proof. Suppose $x \in R_1$ is normal in R. Then R/Rx is an algebra on three generators with three quadratic defining relations, so its point scheme is either $\mathbb{P}^2$ or the zero locus of a nonzero homogeneous cubic polynomial. On the other hand, the point scheme of R/Rx is the intersection of the plane $\mathcal{V}(x)$ with the scheme P. Since $P = Q$, this intersection is a (possibly degenerate) conic, which is a contradiction. ■

We thank M. Van den Bergh for suggesting the use of derivations in the proof of the following result.

Proposition 4.2. *If the point scheme P is the quadric Q, then R is not a twist by an automorphism of a regular algebra that maps onto the homogeneous coordinate ring $S(Q)$ of Q.*

Proof. Suppose $P = Q$. If the result is false, then there exists a regular algebra R' of global dimension four which maps onto $S(Q)$ and has Hilbert series $H_{R'}(t) = (1-t)^{-4}$. The automorphism τ of Q (associated to R') is the identity. We will first show that for such R', the normal element Ω is central.

Since $P \neq \mathbb{P}^3$, and $\tau =$ identity, we may assume R' has generators $y_1, \ldots, y_4$ such that

$$y_3 y_1 = y_1 y_3, \qquad y_3 y_2 = y_2 y_3, \qquad y_1 y_2 - y_2 y_1 = \Omega.$$

Since $P = Q$, the element y_3 is not normal, by Lemma 4.1. Thus we may write $\Omega = y_4y_3 - y_3y_4$. It follows that we may assume that five of the six defining relations of R' have the form:

$$\begin{array}{rclcrcl} y_3y_1 &=& y_1y_3, & & y_4y_3 - y_3y_4 &=& y_1y_2 - y_2y_1, \\ y_3y_2 &=& y_2y_3, & & y_4y_1 - y_1y_4 &=& \mu(y_1y_2 - y_2y_1), \\ & & & & y_4y_2 - y_2y_4 &=& \nu(y_1y_2 - y_2y_1), \end{array}$$

where $\mu, \nu \in k$. A computation yields Ω is central in R'.

Fix $z \in R'_1$, and define $D_z \colon R' \to R'$ by $D_z(a) = \Omega^{-1}(za - az)$. Since Ω is central, D_z is a derivation of R' and so induces a derivation D'_z of $S(Q)$. Since there are no nontrivial derivations of degree -1 of $S(Q)$, it follows that $D'_z \equiv 0$. In particular, $D'_z(y_i) = 0$ for all i, and similarly for D_z. Hence R' is commutative, which contradicts the assumption that $P = Q$. ∎

The proof of Proposition 4.2 shows that R is also not a twist by a twisting system of a regular algebra that maps onto $S(Q)$.

Corollary 4.3. *If $P = Q$, then $\tau \notin Aut(R)$.*

Proof. If $\tau \in \mathrm{Aut}(R)$, then it is possible to twist R by τ^{-1} to an algebra which satisfies the same hypotheses but maps onto $S(Q)$. This is not possible by Proposition 4.2. ∎

These last few results demonstrate that different techniques to those used in §2 need to be employed to deal with the case $P = Q$.

4.2. **Restrictions on the Map τ.**

By Corollaries 3.6 and 4.3, we may assume that if $P = Q$, then $\alpha = \beta \neq \pm\sqrt{-1}$ in Lemma 3.1. We may also assume that $\gamma = 0$ in Lemma 3.1 by taking x_1, $x_2 - \gamma(2\alpha)^{-1}x_1$, x_3, $x_4 - \gamma(2\alpha)^{-1}x_3$ as generators of R. This proves the next lemma.

Lemma 4.4. *If $P = Q$, then there is a choice of generators $x_1, \dots, x_4$ for R such that the defining relations of R may be written*

$$\begin{array}{ll} x_1^\tau x_4 = x_2^\tau x_3, & x_3^\tau x_4 - x_4^\tau x_3 = x_1^\tau x_2 - x_2^\tau x_1, \\ x_1^\tau x_3 = x_3^\tau x_1, & x_1^\tau x_4 - x_4^\tau x_1 = \alpha(x_1^\tau x_2 - x_2^\tau x_1), \\ x_2^\tau x_4 = x_4^\tau x_2, & x_2^\tau x_3 - x_3^\tau x_2 = \alpha(x_1^\tau x_2 - x_2^\tau x_1), \end{array}$$

where $\alpha \in k$, $\alpha(\alpha^2 + 1) \neq 0$. We may take $\Omega = x_2^\tau x_1 - x_1^\tau x_2$ and $Q = \mathcal{V}(x_1x_4 - x_2x_3)$. ∎

As in §3 we take a basis $\eta_1, \dots, \eta_4$ dual to the basis $x_1, \dots, x_4$. It follows from Corollary 3.6(a) that in $S^!$ we have

$$\begin{aligned} \omega &= \eta_4^{\tau^{-1}}\eta_3 + \eta_2^{\tau^{-1}}\eta_1 + \alpha\eta_4^{\tau^{-1}}\eta_1 + \alpha\eta_3^{\tau^{-1}}\eta_2 \\ &\in k^\times(\eta_4^{\tau^{-2}}\eta_3^{\tau^{-1}} + \eta_2^{\tau^{-2}}\eta_1^{\tau^{-1}} - \alpha^{-1}\eta_4^{\tau^{-2}}\eta_1^{\tau^{-1}} - \alpha^{-1}\eta_3^{\tau^{-2}}\eta_2^{\tau^{-1}}) \qquad (*) \end{aligned}$$

and the map $\theta \in \mathrm{Aut}(S^!)$ determined by ω is given by $\theta \in k^\times \phi \circ \tau^{-2}$ where (with respect to the basis $\eta_1, \dots, \eta_4$) ϕ is given by the matrix

$$\phi = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}.$$

Recall from §3 that $\theta \circ \tau \in k\tau \circ \theta$ and $\theta \in \mathrm{Aut}(R)$.

Lemma 4.5. *We have ϕ and $\tau^2 \in Aut(R)$, $\tau \circ \theta = \pm\theta \circ \tau$ and $\tau \circ \phi = \pm\phi \circ \tau$.*

Proof. Since $\theta \circ \tau \in k\tau \circ \theta$, we have $\tau^{-1} \circ \phi = \delta\phi \circ \tau^{-1}$ for some $\delta \in k$. Then

$$\omega^\phi = \delta(\eta_4^{\phi\tau^{-1}}\eta_3^\phi + \eta_2^{\phi\tau^{-1}}\eta_1^\phi + \alpha\eta_4^{\phi\tau^{-1}}\eta_1^\phi + \alpha\eta_3^{\phi\tau^{-1}}\eta_2^\phi) = -\delta\omega.$$

It follows by Lemma 3.3 that $\phi \in \mathrm{Aut}(R)$, so $\tau^2 \in \mathrm{Aut}(R)$ also.

From the definition of θ we have $\omega^{\tau^2} a^{\tau^2} = a^{\theta\tau^2}\omega^{\tau^2}$ for all $a \in S^!$. However $\tau^2 \in \mathrm{Aut}(R^!)$, so $\omega^{\tau^2} \in k^\times\omega$, which implies $\omega a^{\tau^2} = a^{\theta\tau^2}\omega$ for all $a \in S^!$, whence $(a^{\theta\tau^2} - a^{\tau^2\theta})\omega = 0$. Thus $\tau^2 \circ \theta = \theta \circ \tau^2$ since ω is 1-regular by Lemma 3.2. The result follows since $\theta \circ \tau \in k^\times \tau \circ \theta$. ■

Using the second way $(*)$ to write ω, the next result is immediate.

Lemma 4.6. *Using the generators and relations for R given in Lemma 4.4, we have the following alternative defining relations for R:*

$$\begin{aligned} x_1^{\tau^2}x_4^\tau &= x_2^{\tau^2}x_3^\tau, & x_3^{\tau^2}x_4^\tau - x_4^{\tau^2}x_3^\tau &= x_1^{\tau^2}x_2^\tau - x_2^{\tau^2}x_1^\tau, \\ x_1^{\tau^2}x_3^\tau &= x_3^{\tau^2}x_1^\tau, & x_1^{\tau^2}x_4^\tau - x_4^{\tau^2}x_1^\tau &= -\alpha^{-1}(x_1^{\tau^2}x_2^\tau - x_2^{\tau^2}x_1^\tau), \\ x_2^{\tau^2}x_4^\tau &= x_4^{\tau^2}x_2^\tau, & x_2^{\tau^2}x_3^\tau - x_3^{\tau^2}x_2^\tau &= -\alpha^{-1}(x_1^{\tau^2}x_2^\tau - x_2^{\tau^2}x_1^\tau). \end{aligned}$$

Moreover, $\Omega \in k^\times(x_2^{\tau^2}x_1^\tau - x_1^{\tau^2}x_2^\tau)$. ■

To pass from the defining relations for R in Lemma 4.4 to those in Lemma 4.6, we change α to $-\alpha^{-1}$. Applying the same rule again, gives $-\alpha^{-1} \to -(-\alpha^{-1})^{-1} = \alpha$, which agrees with $\tau^2 \in \mathrm{Aut}(R)$. By using the two expressions for Ω from Lemma 4.4 and Lemma 4.6 (and fixing a choice of representative of τ), we find

$$x_1^{\tau^2}\Omega = -\lambda\Omega x_3, \quad x_2^{\tau^2}\Omega = \lambda\Omega x_4, \quad x_3^{\tau^2}\Omega = \lambda\Omega x_1, \quad x_4^{\tau^2}\Omega = -\lambda\Omega x_2$$

for some $\lambda \in k^\times$. Since $\{x_1, \dots, x_4\}$ and $\{\eta_1, \dots, \eta_4\}$ are dual bases, it follows that $a\Omega = \Omega a^\theta$ for all $a \in R$.

According to Lemma 4.5, the next result is a first attempt at determining which τ arise.

Lemma 4.7. *The map ϕ has two (nonintersecting) lines ℓ_1, ℓ_2 of fixed points in $\mathbb{P}^3$, and each ℓ_i meets Q in two distinct points. If $\psi \in Aut(Q)$, then $\psi \circ \phi = \pm\phi \circ \psi$ if and only if $\psi(\ell_1 \cup \ell_2) = \ell_1 \cup \ell_2$. In this case, ψ is given (with respect to the basis $\eta_1, \dots, \eta_4$) by one of* *the following matrices in $GL_4(k)$:*

$$\psi \in k^\times \begin{bmatrix} a & 0 & \mp c & 0 \\ 0 & b & 0 & \mp d \\ c & 0 & \pm a & 0 \\ 0 & d & 0 & \pm b \end{bmatrix} \qquad \text{or} \qquad \psi \in k^\times \begin{bmatrix} 0 & a & 0 & \pm c \\ b & 0 & \pm d & 0 \\ 0 & c & 0 & \mp a \\ d & 0 & \mp b & 0 \end{bmatrix} \tag{1}$$

where $ad = bc$ and $ab + cd \neq 0$, or

$$\psi \in k^\times \begin{bmatrix} a & b & \pm ia & \pm ib \\ c & d & \mp ic & \mp id \\ -ia & ib & \pm a & \mp b \\ -ic & id & \mp c & \pm d \end{bmatrix} \tag{2}$$

where $ad = bc$ and $i^2 = -1$. In (1), ψ preserves the rulings on Q, but in (2), ψ interchanges the rulings. ■

By Corollary 3.6 we know that the element

$$\omega' = \tau^{-2}(\eta_4\eta_3^\tau + \eta_2\eta_1^\tau - \alpha^{-1}\eta_4\eta_1^\tau - \alpha^{-1}\eta_3\eta_2^\tau)$$

is a nonzero scalar multiple of ω. We use this fact combined with Lemma 4.7 to impose necessary conditions on τ in order to find which algebras, with defining relations given by those in Lemma 4.4, are regular.

Proposition 4.8. *Let $A = k[x_1, \dots, x_4]$ be a k-algebra with defining relations given by those in Lemma 4.4 where $\alpha(\alpha^2+1) \neq 0$, $\tau \in Aut(Q)$ and $\tau \circ \phi = \pm\phi \circ \tau$. Then $\tau^2 \in Aut(A)$ and $\Omega = x_2^\tau x_1 - x_1^\tau x_2$ is normal in A. Moreover, $\omega' \in k^\times\omega$ and ω is normal in $S^!$ if and only if either*

(a) τ interchanges the rulings on Q and commutes with ϕ, or

(b) τ preserves the rulings on Q, $\tau \circ \phi = -\phi \circ \tau$ and $\alpha^2 = 1$.

If (a) or (b) hold, then A also has defining relations given by those in Lemma 4.6, $\Omega \in k^\times(x_2^{\tau^2}x_1^\tau - x_1^{\tau^2}x_2^\tau)$ and is 1-regular, and $\theta \in k^\times\phi \circ \tau^{-2}$ is determined on A by $a\Omega = \Omega a^\theta$ for all $a \in A_1$.

Proof. Although A is conceivably not regular, Lemma 1.5 and Lemma 3.3 still apply so Ω is normal and $A^! = S^!/\langle\omega\rangle$ where $\omega \in S_2^!$ is 1-regular (but ω may not be normal). Since $\tau \in \text{Aut}(Q)$, it is an automorphism of S and $S^!$, so it suffices to prove $\omega^{\tau^2} \in k^\times\omega$. By hypothesis $\tau \circ \phi = \pm\phi \circ \tau$, so τ^2 preserves the rulings on Q and commutes with ϕ. Hence τ^2 may be represented by a matrix in (1) of Lemma 4.7 (with upper sign), and computing ω^{τ^2} with such a matrix gives $\omega^{\tau^2} \in k^\times\omega$. Moreover, $\omega a = b(\omega')^{\tau^2}$ where $a, b \in S_1^!$, and a computation with the matrices in Lemma 4.7 yields $\omega' \in k^\times\omega$ if and only if (a) or (b) hold. In this situation, Ω is 1-regular by [16, Lemma 2.5], and since $\omega^\phi \in k^\times\omega$, we have $\phi \in \text{Aut}(A)$ by Lemma 3.3, so $\theta \in \text{Aut}(A)$ too. ■

Some of the algebras determined by the hypotheses of Proposition 4.8 together with condition (b) were found in [21]. We will prove in the next subsection that if the hypotheses of Proposition 4.8 are satisfied (including conditions (a) or (b)) then such an algebra satisfies the hypotheses given in §1.1 for our algebras R and hence classify all such R with $P = Q$. The algebras determined by condition (a) have not appeared elsewhere; in particular, they did not arise in [21] since the map τ interchanges the rulings on Q.

Lemma 4.9. *If A satisfies the hypotheses of Proposition 4.8, including conditions (a) or (b), then ω is regular in $S^!$ and $H_{A^!}(x) = (1+x)^4$.*

Proof. By Proposition 4.8, ω is normal and $\omega' \in k^\times \omega$. Since $H_{S^!}(x) = (1-x)^{-1}(1+x)^3$, it follows that $H_{A^!}(x) \geq (1+x)^4$, with equality holding if and only if ω is regular in $S^!$. On the other hand, a basis of $S^!$ consists of (at most) monomials of the form

$$\tau^n(\eta_1^{i_1}(\eta_2^{\tau}\eta_3^{\tau^2}\eta_2^{\tau^3}\cdots)^{i_2}(\eta_4^{\tau^m})^{i_3}) \qquad \text{and} \qquad \tau^n(\eta_1^{i_1}(\eta_3^{\tau}\eta_2^{\tau^2}\eta_3^{\tau^3}\cdots)^{i_2}(\eta_4^{\tau^m})^{i_3})$$

for all $n \in \mathbb{Z}$ where $i_1, i_2, i_3 \in \{0,1\}$, and in passing to $A^!$ we may (at least) eliminate multiples of $\tau^{2n}(\eta_2\eta_3^{\tau})$ and $\tau^{2n}(\eta_2^{\tau}\eta_3^{\tau^2})$ for all $n \in \mathbb{Z}$ (since $\omega = \omega' = 0$ in $A^!$). It follows that $H_{A^!}(x) \leq (1+x)^4$, which completes the proof. ■

4.3. **Classification of the Algebras R with Point Scheme $P = Q$.** We use the notion of a twisting system introduced in [23] to twist an algebra given by (a) or (b) in Proposition 4.8 to an algebra which is more amenable to study.

Definition 4.10. [23] Let $A = \bigoplus_{n\in\mathbb{Z}} A_n$ be a $\mathbb{Z}$-graded k-algebra. A set $\{t_n : n \in \mathbb{Z}\}$ of graded k-linear bijections of A is called a twisting system of A if $t_n(yt_m(z)) = t_n(y)t_{n+m}(z)$ or, equivalently, if

$$t_n(yz) = t_n(y)t_{n+m}t_m^{-1}(z)$$

for all $y \in A_m$, $z \in A_i$. The binary map $*$ defined by $y * z = yt_m(z)$ for all $y \in A_m$, $z \in A_i$ defines a new graded, associative multiplication $*$ on $\bigoplus_{n\in\mathbb{Z}} A_n$ giving an algebra called the twist of A by the twisting system $\{t_n\}$. It is proved in [23] that regularity, Hilbert series, the property of being Koszul, etc. are invariant properties under twisting.

Remark 4.11. In the case of our quadratic algebras R, suppose $\{t_n\}$ is a twisting system of R where $t_1 \in \mathrm{Aut}(Q)$ and let $t = t_1 \circ \tau$ on Q. Then, for example, $x_1 * x_3^{t^{-1}} = x_1 x_3^{t^{-1}t_1} = x_1 x_3^{\tau^{-1}} = x_3 x_1^{\tau^{-1}} = x_3 * x_1^{t^{-1}}$. Similarly for the other five relations of R, so that the twist of R by $\{t_n\}$ has point scheme $(Q,t) = (Q, t_1 \circ \tau)$ and the parameter α is unchanged.

Henceforth, let $A(\alpha,\tau) = T(V)/\langle W \rangle$ denote an algebra whose defining relations are given by those in Lemma 4.4. The twisting system given in the proof of the following result is the simplest one we could find to twist our candidates for R to one particular algebra.

Proposition 4.12. *Suppose that $\alpha(\alpha^2+1) \neq 0$. If either*
(a) τ interchanges the rulings on Q and commutes with ϕ, or
(b) τ preserves the rulings on Q, $\tau \circ \phi = -\phi \circ \tau$ and $\alpha^2 = 1$,
then $A(\alpha,\tau)$ twists by a twisting system to an algebra $A(\alpha,t)$ where $t \in Aut(Q)$ has matrix representative

$$t = \frac{1}{2}\begin{bmatrix} 1 & 1 & i & i \\ 1 & 1 & -i & -i \\ -i & i & 1 & -1 \\ -i & i & -1 & 1 \end{bmatrix}$$

(with respect to the basis $\eta_1, \dots, \eta_4$) where $i^2 = -1$.

Proof. Notice that $t^2 = 1$ and $t \circ \phi = \phi \circ t$. We define a potential twisting system of $A(\alpha,\tau)$ as follows. Let

$$t_n = \begin{cases} \tau^{-n} & \text{if } n \in 2\mathbb{Z} \\ \tau^{-n+1} \circ t_1 & \text{if } n \in 2\mathbb{Z}+1 \end{cases}$$

where $t_1 : A(\alpha,\tau)_m \to A(\alpha,\tau)_m$ is the linear map given by

$$t_1(a_1 \cdots a_m) = a_1^{t\tau^{-1}} a_2^{\tau(t\tau^{-1})\tau^{-1}} \cdots a_m^{\tau^{m-1}(t\tau^{-1})\tau^{-(m-1)}}$$

where $a_i \in A(\alpha,\tau)_1$ for all i. By Proposition 4.8, we have $\tau^2 \in \mathrm{Aut}(A(\alpha,\tau))$, so to prove that t_n is well defined, it suffices to prove that t_1 is well defined. Viewing t_1 as a linear map on $T(V)$, we have that t_1 is bijective, and since $t^2 = 1$ we have

$$t_1(V^i ab V^j) \subset \begin{cases} V^i(\tau^{-i} t_1 \tau^i(ab))V^j & \text{if } i \in 2\mathbb{N} \cup \{0\} \quad (1) \\ V^i(\tau^{-i-1} t_1^{-1} \tau^{i-1}(ab))V^j & \text{if } i \in 2\mathbb{N}-1, \quad (2) \end{cases}$$

for all $a,b \in T(V)_1$, for all $j \in \mathbb{N}$. Thus it suffices to prove $t_1(W) = W$, which is equivalent to proving $(\tau^2 \circ t_1)(W) = W$. But $(\tau^2 \circ t_1)(ab) = a^{t\tau} b^{\tau t}$, so $\tau^2 \circ t_1(x_1^\tau x_4 - x_2^\tau x_3)$ and $\tau^2 \circ t_1(x_i^\tau x_j - x_j^\tau x_i)$ vanish on $\Gamma_\tau(Q)$ for all i,j. It follows that $\tau^2 \circ t_1$ maps S_2 to S_2 and also $S_2^!$ to $S_2^!$ and is defined on $S_2^!$ by $(\tau^2 \circ t_1)(\eta\eta') = \eta^{\tau t}(\eta')^{t\tau}$ for all $\eta, \eta' \in S_1^!$. A computation shows that $(\tau^2 \circ t_1)(\omega) \in k^\times \omega'$, and $\omega' \in k^\times \omega$ by Proposition 4.8. It follows that $t_1(W) = W$.

To show that $\{t_n\}$ is a twisting system we need to check that

$$t_n(ab) = t_n(a) t_{n+m} t_m^{-1}(b)$$

for all $a \in A(\alpha,\tau)_m$, $b \in A(\alpha,\tau)_i$. This is trivial if $n \in 2\mathbb{Z}$ since $\tau^2 \in \mathrm{Aut}(A(\alpha,\tau))$, so we will assume $n \in 2\mathbb{Z}+1$. If $m \in 2\mathbb{Z}$, then

$$\begin{aligned} t_n(a) t_{n+m} t_m^{-1}(b) &= \tau^{-n+1}(t_1(a) \tau^{-m} t_1 \tau^m(b)) \\ &= \tau^{-n+1} t_1(ab) \\ &= t_n(ab) \end{aligned}$$

where the second equality follows from (1). If $m \in 2\mathbb{Z}+1$, then

$$\begin{aligned} t_n(a) t_{n+m} t_m^{-1}(b) &= \tau^{-n+1}(t_1(a) \tau^{-m-1} t_1^{-1} \tau^{m-1}(b)) \\ &= t_n(ab) \end{aligned}$$

as before by using (2).

As an automorphism of Q, we have $t_1 = t \circ \tau^{-1}$, so $t_1 \circ \tau = t$. It follows from Remark 4.11 that the twist of $A(\alpha, \tau)$ by $\{t_n\}$ is $A(\alpha, t)$. ■

A study of $A(\alpha, t)$ will follow in [22] where t is the map given in Proposition 4.12. In particular, it is shown in [22] that $A(\alpha, t)$ is a finite module over its centre and has no normal elements of odd degree. Moreover the map t has four eigenvectors in $A(\alpha, t)_1$, the squares of which are central elements. At this stage, it is conceivable that $A(\alpha, t)$ is not regular for generic values of α, so that no such R with $P = Q$ exists; fortunately, our next result proves it is regular.

Theorem 4.13. *The regular algebra R whose point scheme P is the quadric Q is isomorphic to the algebra $A(\alpha, \tau) = k[x_1, \dots, x_4]$ with defining relations*

$$\begin{array}{ll} x_1^\tau x_4 = x_2^\tau x_3, & x_3^\tau x_4 - x_4^\tau x_3 = x_1^\tau x_2 - x_2^\tau x_1, \\ x_1^\tau x_3 = x_3^\tau x_1, & x_1^\tau x_4 - x_4^\tau x_1 = \alpha(x_1^\tau x_2 - x_2^\tau x_1), \\ x_2^\tau x_4 = x_4^\tau x_2, & x_2^\tau x_3 - x_3^\tau x_2 = \alpha(x_1^\tau x_2 - x_2^\tau x_1), \end{array}$$

where $\alpha \in k$, $\alpha(\alpha^2+1) \neq 0$ and (with respect to the basis η_i dual to the x_i) $\tau \in Aut(Q)$ is given by either

$$(a) \qquad \tau \in k^\times \begin{bmatrix} a & b & ia & ib \\ c & d & -ic & -id \\ -ia & ib & a & -b \\ -ic & id & -c & d \end{bmatrix} \qquad \text{where } i^2 = -1 \text{ and } a, \dots, d \in k \text{ with } ad = bc,$$

or

$$(b) \qquad \tau \in k^\times \begin{bmatrix} a & 0 & c & 0 \\ 0 & b & 0 & d \\ c & 0 & -a & 0 \\ 0 & d & 0 & -b \end{bmatrix} \qquad \text{or} \qquad \tau \in k^\times \begin{bmatrix} 0 & a & 0 & -c \\ b & 0 & -d & 0 \\ 0 & c & 0 & a \\ d & 0 & b & 0 \end{bmatrix}$$

providing $\alpha^2 = 1$, where $a, \dots, d \in k$ with $ad = bc$ and $ab + cd \neq 0$.
For both types of τ, we have $Q = \mathcal{V}(x_1x_4 - x_2x_3)$.
Conversely, such an $A(\alpha, \tau)$ which satisfies (a) or (b) is a regular algebra R satisfying the conditions of §1.1 and has point scheme equal to the quadric Q.

Proof. The first statement follows from Lemma 4.4, Lemma 4.7 and Proposition 4.8. To prove the converse, it suffices (by Proposition 4.12 and [23]) to prove regularity and Hilbert series of only one $A(\alpha, \tau)$ in the statement; we use an $A(\alpha, \tau)$ in which Ω is central.

Let $A = A(\alpha, T)$ where (with respect to the basis $\eta_1, \dots, \eta_4$) the map T has representative

$$T = \begin{bmatrix} 1 & 1 & i & i \\ -1 & -1 & i & i \\ -i & i & 1 & -1 \\ i & -i & 1 & -1 \end{bmatrix}$$

and assume $\alpha(\alpha^2+1) \neq 0$. Thus, T determines a matrix from (a) in the statement. With this choice of representative for T, we have $T^2 = -4i\phi$ and $\theta = \lambda\phi \circ T^{-2} = \lambda i/4$ where $i^2 = -1$ and $\lambda \in k^\times$ (λ depends on the choice of matrix representing T). It follows that

$\omega = 4i\alpha\omega'$ and so $\omega\eta_1^T = -4i\eta_3^{T^{-1}}\omega$. Thus, $\eta_1^\theta = -4i\eta_3^{T^{-2}} = \eta_3^{-\phi} = \eta_1$. But θ is a scalar multiple of the identity, hence $\theta = 1$ and Ω is central in A.

It follows that $A^! = S^!/\langle\omega\rangle$ where ω is central. By [14], $S(Q)$ is a Koszul algebra, so S and $S^!$ are also (by [23]). Since ω is regular by Lemma 4.9, we may apply [14] to $A^! = S^!/\langle\omega\rangle$ to conclude that $A^!$, and hence A, is a Koszul algebra. Then $H_A(x)$ is given by the formula $H_A(x)H_{A^!}(-x) = 1$, so by Lemma 4.9, $H_A(x) = (1-x)^{-4}$, and Ω is regular in A. By [10], it follows that A is Auslander-Gorenstein (hence Gorenstein), so to prove regularity of A it suffices to prove $\text{gldim}(A) = 4$. Since A is a Koszul algebra and $H_{A^!}(x) = (1+x)^4$, the trivial module $k = A/Ax_1 + \cdots + Ax_4$ has a minimal (free) resolution of length four, so $\text{gldim}(A) = 4$. Hence, the point scheme of A is well defined by Theorem 1.10, and the conditions on α guarantee that $T \notin \text{Aut}(A)$, so the point scheme of A is Q. ■

The line modules over the algebras $R = A(\alpha, \tau)$ in Theorem 4.13 will be classified in [22]. By [11], the isomorphism classes of left line modules are in one-to-one correspondence with the isomorphism classes of right line modules. However, in contrast with the case $P \neq Q$ and other examples (such as the Sklyanin algebra [11] and homogenizations of enveloping algebras [7, 9]), it is possible for a line in $\mathbb{P}^3$ which corresponds to a left line module over R not to correspond to a right line module. For example, by Lemma 4.4 and [11], the line $\ell = \mathcal{V}(x_4 - \alpha x_2, x_1) \not\subset Q$ corresponds to a left line module, but by Lemma 4.6 it does not correspond to a right line module since $\alpha^2 \neq -1$ (but $\tau^{-1}(\ell)$ does). This suggests that the algebras R, which are not determined by their point scheme, are perhaps "determined" by their line modules and associated automorphism (their "line scheme").

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