EMBEDDING A QUANTUM RANK THREE QUADRIC IN A QUANTUM $\mathbb{P}^3$

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ABSTRACT. We continue the classification, begun in [11], [14] and [12], of quadratic Artin-Schelter regular algebras of global dimension 4 which map onto a twisted homogeneous coordinate ring of a quadric hypersurface in $\mathbb{P}^3$. In this paper, we consider those cases where the quadric has rank 3. We also give sufficient conditions for the point scheme of any quadratic regular algebra of global dimension 4 to be the graph of an automorphism.

INTRODUCTION

In [2], a notion of regularity, here called Artin-Schelter regularity, was introduced for non-commutative graded algebras. A classification and a general analysis of Artin-Schelter regular algebras of global dimension 3 were carried out in [2], [3] and [4]. A classification theorem for Artin-Schelter regular algebras of global dimension 4 has proved to be much less tractable and is still a long way off. In [11], [12] and [14], the Artin-Schelter regular algebras of global dimension four which map onto a twisted homogeneous coordinate ring of a nonsingular quadric in $\mathbb{P}^3$ were classified; in this paper, a similar problem is addressed for rank three quadrics in $\mathbb{P}^3$.

Throughout, k is an algebraically closed field of characteristic different from two, all algebras are k-algebras and all schemes are defined over k. Let Q be a quadratic hypersurface in $\mathbb{P}^3$ and let τ be an automorphism of Q. Associated to the pair (Q, τ) there is a twisted homogeneous coordinate ring $S = S(Q, \tau)$ ([3]). Let $\mathcal{R}(Q, \tau)$ be the family of connected, quadratic, Artin-Schelter regular algebras of global dimension 4 which have S as a graded factor ring. If $R \in \mathcal{R}(Q, \tau)$, then, in the language of [1], $\text{Proj}R$ may be viewed as a quantum $\mathbb{P}^3$. As such, we seek those quantum $\mathbb{P}^3$s for which $\text{Proj}S \hookrightarrow \text{Proj}R$.

In [12], the families $\mathcal{R}(Q, \tau)$ are classified whenever the quadric Q is nonsingular. Describing the results from [12] requires the notion of the point scheme of an algebra $R \in \mathcal{R}(Q, \tau)$. Let A be a graded algebra generated by four elements of degree one and defined by six quadratic

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relations. Let Γ_2 be the scheme in $\mathbb{P}^3 \times \mathbb{P}^3$ of the zeros of the six relations and let P be the projection of Γ_2 onto the first $\mathbb{P}^3$. The first main result of [12], namely [12, Theorem 1.10], gives a set of conditions on A sufficient to guarantee that Γ_2 is the graph of an automorphism σ of P. The algebras in $\mathcal{R}(Q, \tau)$ satisfy these sufficient conditions. The scheme Γ_2, or equivalently the pair (P, σ), is called the point scheme of the algebra A, since it parametrizes the isomorphism classes of point modules of A.

If $R \in \mathcal{R}(Q, \tau)$, then $Q \subset P$ and $\sigma|_Q = \tau$. Thus the classification of R, for Q nonsingular, splits into two cases, depending on whether $P \neq Q$ or $P = Q$. In the first case it is proved in [12] that the defining relations of R can be recovered directly from the point scheme (P, σ) as the bihomogeneous two-forms which vanish on Γ_2. This theorem reduces the classification to finding the allowable pairs (P, σ). In the second case, where $P = Q$, the analysis is more involved because the defining relations of R cannot be recovered from the point scheme (P, σ). These cases are analyzed using Koszul duality and the notion of twisting sytems from [15]. The reader is referred to [12] for the precise statements of the classification when Q is nonsingular.

The structure of this paper roughly models that of [12]. In Section 1 we include some general results about Artin-Schelter regular algebras. In particular, in Theorem 1.4, we prove a result analogous to [12, Theorem 1.10] discussed above concerning the point scheme being the graph of an automorphism, but without the assumption of a certain technical hypothesis of [12, Theorem 1.10].

In the remaining two sections of the paper, the sets $\mathcal{R}(Q, \tau)$ are analyzed whenever the quadric Q has rank 3; that is, whenever the quadratic form corresponding to Q has rank 3. In particular, in this setting, the quadric Q has a unique singular point through which all lines on the quadric pass. As in [12], the point scheme (P, σ) plays a critical role, and the analysis splits into two cases: $P \neq Q$ and $P = Q$. In Section 2, a classification of the algebras in $\mathcal{R}(Q, \tau)$ for which $P \neq Q$ is given in Theorem 2.8. There are five types which occur, and every such algebra is a twist by τ of an algebra in $\mathcal{R}(Q, \mathrm{id})$, c.f., Lemma 2.1. The remaining case, namely where $P = Q$, is addressed in Section 3. Here, the automorphisms τ which arise are severely limited and are given in Lemma 3.7. As such, there are only two types of algebras in $\mathcal{R}(Q, \tau)$ for which $P = Q$, and they are classified in Theorem 3.10.

1. General Remarks on Regular Algebras of Global Dimension 4

Throughout, k denotes an algebraically closed field such that $\mathrm{char}(k) \neq 2$. We first collect some general results on Artin-Schelter regular algebras of global dimension 4 and the scheme of point modules for such algebras.

1.1. **Definitions.**

Throughout the paper A denotes an $\mathbb{N}$-graded algebra of the form $A = T(V^*)/I$ where V^* is the k-dual of a finite dimensional vector space V, $T(V^*)$ is the tensor algebra on V^* and I is a finitely generated graded ideal with $I_1 = 0$. Such algebras are connected, and generated by degree one elements. For any bounded below $\mathbb{Z}$-graded k-space M, the Hilbert Series of M is $H_M(t) := \sum_{n\in\mathbb{Z}} \dim(M_n)t^n$.

Definition 1.1. [2] An algebra A as above is called Artin-Schelter regular (of dimension d) if
(a) A has finite global homological dimension, $\mathrm{gldim}(A) = d$,
(b) A has finite Gelfand-Kirillov dimension,
(c) A is Gorenstein; that is, $\mathrm{Ext}^q_A(k, A) = \delta^q_d k$.

Throughout the paper the term regular will mean Artin-Schelter regular.

The algebra A is quadratic if its ideal I of relations is generated by degree two elements. The Koszul dual of such an algebra is defined to be the quadratic algebra $A^! := T(V)/\langle I_2^\perp\rangle$ where $I_2^\perp \subset V \otimes V$ is the orthogonal space to $I_2 \subset V^* \otimes V^*$.

Definition 1.2. [5] The algebra A is said to be Koszul if its dual algebra $A^!$ is isomorphic to the cohomology algebra $\mathrm{Ext}^*(k, k)$.

It is known that there are many different criteria which are equivalent to the above definition of Koszul. For lack of a suitable reference, we include a proof of the following well known result.

Lemma 1.3. *If A is Artin-Schelter regular of global dimension 4 with four linearly independent generators, then A is Koszul and has Hilbert Series $H_A(t) = (1-t)^{-4}$; in particular, A is quadratic with six defining relations and has Gelfand-Kirillov dimension four.*

Proof. By hypothesis, $\dim(V) = 4$. Fix a basis $x_1, \dots , x_4$ for V^* and put $x = (x_1, \dots , x_4)^t$. Choose a minimal set of graded generators $f_1, f_2, \dots , f_n$ of degrees $r_1, r_2, \dots , r_n$ respectively for the ideal I. Put $f = (f_1, f_2, \dots , f_n)^t$ and define an $n\times 4$ matrix M of homogeneous tensors by $M\otimes x = f$. The hypothesis that A has global dimension four and the Gorenstein symmetry property ensure the existence of a minimal, graded, projective resolution of the right trivial A-module k_A of the form

$$0 \to A[-d] \xrightarrow{x^t} A^4[-d+1] \xrightarrow{N} \bigoplus_{1\le i\le n} A[-r_i] \xrightarrow{M} A^4[-1] \xrightarrow{x} A \to k \to 0 \qquad (R1)$$

for some $d \geq 4$ and some $4 \times n$ matrix N of homogeneous tensors. The dual of this exact sequence, augmented by ${}_Ak[d]$, is a minimal projective resolution of ${}_Ak[d]$ and so the entries of the vector of homogeneous tensors $x^t \otimes N$ form a generating set for the ideal I. In particular, the degrees r_i of the generators f_i are the same as those of the entries of $x^t \otimes N$, the latter degrees being $d-r_1, d-r_2, \dots, d-r_n$. Thus, the set of degrees $\{r_1, \dots, r_n\}$ is invariant under the map $r \mapsto d-r$.

Let $p(t) = 1-4t+\sum_i t^{r_i} - 4t^{d-1} + t^d$. From the resolution (R1), we have that $H_A(t) = p(t)^{-1}$. Since A has polynomial growth, $p(t)$ has no positive real roots less than one. Combining this with $p(0) = 1$ and $p(1) = 0$, it follows that either $p'(1) > 0$, or both $p'(1) = 0$ and $p''(1) \geq 0$. However, $0 = p(1) = n-6$, whence $n = 6$, and $p'(1) = -4 + \sum_i r_i - 4(d-1) + d = \sum_i r_i - 3d$. Since $\sum_i r_i = \sum_i (d-r_i)$, we obtain $\sum_i r_i = nd/2$, which equals $3d$ since $n = 6$. Hence $p'(1) = 0$. Therefore, $0 \leq p''(1) = \sum_i r_i^2 - 3d^2 + 8d - 8$. The invariance of $\{r_1, \dots, r_6\}$ under the map $r \mapsto d-r$ implies that $\sum_i r_i^2$ is maximized by taking $\{r_1, \dots, r_6\} = \{d-2, d-2, d-2, 2, 2, 2\}$. It follows that the integer d, which is strictly greater than three, satisfies the inequality $3d^2 - 8d + 8 \leq 3(d-2)^2 + 12$. Hence $d = 4$. Thus, $r_i = 2$ for all i, and so $p(t) = 1 - 4t + 6t^2 - 4t^3 + t^4 = (1-t)^4$. This proves the Hilbert series claim.

By the above analysis, the matrices M and N consist entirely of one-forms. Therefore, the maps in (R1) are made up of linear terms, which is one of the equivalent definitions of Koszul given in [5]. Hence, A is a Koszul algebra. ■

We remark that the converse of Lemma 1.3 is false; a counterexample is provided in Example 1.6.

If A is an algebra satisfying the hypotheses of Lemma 1.3, then the proof of the lemma yields that a minimal projective resolution of the trivial graded A-module k_A may be written as

$$0 \to A[-4] \xrightarrow{x^t} A^4[-3] \xrightarrow{N} A^6[-2] \xrightarrow{M} A^4[-1] \xrightarrow{x} A \to k \to 0 \qquad (R2)$$

using the same notation as in the proof of Lemma 1.3.

It is easy to generalize Lemma 1.3 to find all possible configurations of numbers of generators, degrees of relations and Hilbert series of Artin-Schelter regular algebras of global dimension 4.

1.2. **Point Schemes.**

It is well known that the zero locus in $\mathbb{P}^3 \times \mathbb{P}^3$ of the defining relations of an algebra with four generators and six quadratic defining relations is nonempty. To see this, consider the Segre embedding from $\mathbb{P}^3 \times \mathbb{P}^3$ to $\mathbb{P}^{15}$. Under this embedding the zero locus is isomorphic to the intersection in $\mathbb{P}^{15}$ of a 9-dimensional linear variety and the 6-dimensional scheme consisting of the projectivization of the rank one 4×4 matrices. Such an intersection has dimension bigger than or equal to zero (and generically has dimension zero), whence the zero locus is nonempty.

As discussed extensively in [7], many algebras satisfy certain regularity conditions stronger than Artin-Schelter regularity. In this subsection, we show that the zero locus in $\mathbb{P}^3 \times \mathbb{P}^3$ of the defining relations of such an algebra is the graph of an automorphism. Our theorem improves [12, Theorem 1.10]. In the next subsection, we establish that the algebras to be classified herein satisfy the stronger regularity conditions, which are the following.

1. The Hilbert series of the algebra A is $H_A(t) = (1-t)^{-4}$.
2. The algebra A is Noetherian and Auslander-regular of global dimension 4.
3. The algebra A satisfies the Cohen-Macaulay property.

Theorem 1.4. *Suppose $A = T(V^*)/I$ and let $\Gamma_2 \subset \mathbb{P}(V) \times \mathbb{P}(V)$ be the scheme of zeros of $I_2 \subset V^* \otimes V^*$. For $i = 1, 2$, let $\pi_i : \mathbb{P}(V) \times \mathbb{P}(V)$ be the i'th coordinate projection to $\mathbb{P}(V)$ and set $P_i = \pi_i(\Gamma_2)$. If A satisfies conditions 1-3 above, then $P_1 = P_2$, $\pi_i : \Gamma_2 \to P_i$ is a scheme isomorphism for each i, and Γ_2 defines the graph of a scheme automorphism $\sigma : P_1 \to P_2$.*

Proof. By the preceding discussion, Γ_2 is nonempty. Theorem 1.10 of [12] guarantees that the result holds if one technical condition is satisfied: each of P_1 and P_2 contain at least two distinct closed points. So let us assume that the technical condition fails; that is, one of P_1 or P_2 contains only one closed point. For the sake of argument, we take P_1 to have only one closed point $p \in \mathbb{P}(V)$, the other case being similar. As in the proof of Theorem 1.10 of [12], it suffices to prove that the two projections $\pi_i : \Gamma_2 \to P_i$ are injective on closed points. By assumption, $\Gamma_2 = \pi_1^{-1}(P_1)$, so we are reduced to proving that Γ_2 itself has only one closed point.

By [7], the algebra A is an Artin-Schelter regular algebra of global dimension 4. Thus, by Lemma 1.3, A is quadratic with four generators and six defining relations. As in the proof of Lemma 1.3, let $x = (x_1, \ldots, x_4)^t$ where $x_1, \ldots, x_4$ is a basis for V^*. Let M and N be the matrices of one-forms on V defined in the projective resolution (R2). By the Gorenstein symmetry property, the entries of $M \otimes x^t$ form a basis for the space of quadratic relations I_2, and similarly for the entries of $x \otimes N$. Suppose that Γ_2 has two distinct closed points (p, q) and (p, r), where $q, r \in \mathbb{P}(V)$. Then one of q or r is not equal to p, say $q \neq p$. In particular, $q \notin P_1$ and so, for all $s \in \mathbb{P}(V)$, $(q, s) \notin \Gamma_2$. This implies that $M(q)s \neq 0$ for all $s \in \mathbb{P}(V)$ or, equivalently, that the rank of $M(q) = 4$. On the other hand, by the resolution (R2), the entries of the matrix of two-forms $N \otimes M$ vanish in A; that is, the entries of $N \otimes M$ belong to I_2. Since $(p, q) \in \Gamma_2$, it follows that $N(p)M(q) = 0$, so that the rank of $N(p)$ is at most 2. Thus, there exist distinct closed points $u, v \in \mathbb{P}(V)$ such that $uN(p) = vN(p) = 0$, so that $u \in P_1$ and $v \in P_1$, which is a contradiction. Hence, Γ_2 has only one closed point, as required. ■

In the notation of Theorem 1.4, we denote P_1 by P whenever $P_1 = P_2$.

Remark 1.5. By [3], if Γ_2 is the graph of an automorphism, then the set of closed points of Γ_2, or equivalently of P, parametrizes the set of isomorphism classes of point modules of A. Hence, following [12], we refer to P as the "point scheme" of A. It is possible for the scheme

Γ_2 to have only one closed point, in which case the point has multiplicity 20. Some examples of this phenomenon are discussed in [9] and [13].

1.3. **Regular algebras associated to a Quadric Q.**

Let Q denote a rank three quadric in $\mathbb{P}^3$ and let $S(Q)$ (respectively, $S(\mathbb{P}^3)$) denote the (commutative) homogeneous coordinate ring of Q (respectively, $\mathbb{P}^3$). We fix an automorphism τ of Q. Since the class group of Q is isomorphic to $\mathbb{Z}$ and since the Picard group of Q, as embedded in the class group of Q, is isomorphic to $2\mathbb{Z}$, a similar argument to that in [6] to prove that every automorphism of $\mathbb{P}^n$ is linear establishes that τ is linear. Since the span of the points on Q is $\mathbb{P}^3$, it follows that τ is the restriction of a unique automorphism of $\mathbb{P}^3$ (which we also denote by τ). Hence, τ is defined on $S(Q)_1$ via $\tau(x) = x \circ \tau$ for all $x \in S(Q)_1$, and τ may be extended to an automorphism of $S(Q)$ in the obvious way. We write x^τ for $\tau(x)$. Let S denote the twist of $S(Q)$ by τ. Then S is the twisted homogeneous coordinate ring $B(Q, \tau, \mathcal{O}_Q(1))$ ([11, §3]). Moreover, any twist of $S(Q)$ (respectively, of $S(\mathbb{P}^3)$) by a twisting system is isomorphic to a twist of $S(Q)$ (respectively, $S(\mathbb{P}^3)$) by an automorphism.

Let $\mathcal{R}(Q, \tau)$ be the set of all graded algebras R such that R is a quadratic, Artin-Schelter regular algebra of global dimension four with Hilbert series $H_R(t) = (1-t)^{-4}$ and such that there is a graded, degree zero, onto homomorphism $R \twoheadrightarrow S$. (In particular, in the language of [1, 14], $\operatorname{Proj} S$ embeds in the quantum space $\operatorname{Proj} R$.) Henceforth, R denotes an algebra in some appropriate $\mathcal{R}(Q, \tau)$.

Since the algebra S is quadratic, its defining relations are bihomogeneous forms which vanish on the graph of τ. Thus, since R is quadratic and $H_S(t) = (1+t)(1-t)^{-3}$, there exists $\Omega \in R_2$ such that $S \cong R/\langle \Omega \rangle$. There are no nonzero elements of S_2 that vanish on the graph of τ, so any nonzero element of R_2 that vanishes on the graph of τ is a scalar multiple of Ω. In particular, if $u, v \in R_1$, then $u^\tau v - v^\tau u \in R_2$ and is a scalar (possibly zero) multiple of Ω.

In [12], where the quadric has rank four, the quadratic algebras A which have an element $\Omega \in A_2$ such that $A/\langle \Omega \rangle = S$, have the property that Ω is normal in A. The following example demonstrates that in general this is false if the quadric has rank three.

Example 1.6. Let $A = k[x_1, \ldots, x_4]$ with defining relations

$$\begin{aligned} x_1^2 &= x_2 x_3, & \qquad x_1 x_4 &= x_4 x_1, \\ x_1 x_2 &= x_2 x_1, & x_2 x_4 &= x_4 x_2, \\ x_1 x_3 &= x_3 x_1, & x_3 x_4 &= x_4 x_3. \end{aligned}$$

The quotient $A/\langle \Omega \rangle$, where $\Omega = x_2 x_3 - x_3 x_2$, is the homogeneous coordinate ring of a rank three quadric in $\mathbb{P}^3$. However, A is not noetherian, so Ω is not normal in A by [3, Lemma 8.2]. We remark that A is not a regular algebra; in fact, it is a polynomial extension of A/Ax_4 which is a non-Gorenstein, Koszul algebra with Hilbert series $H(t) = (1-t)^{-3}$ and which has global dimension three. (In particular, A/Ax_4 is not a domain, since $\Omega x_3 = 0$ in A.) It follows that A is a non-Gorenstein, Koszul algebra with Hilbert series $H(t) = (1-t)^{-4}$ and has global dimension four.

Lemma 1.7. *If A is a quadratic, regular algebra with an element $\Omega \in A_2$ such that $A/\langle\Omega\rangle = S$, then Ω is normal in A.*

Proof. By an argument similar to that in [12, Lemma 1.5], we may assume that $S = S(Q)$ and that $\tau =$ identity, so that $uv - vu \in k\Omega$ for all $u, v \in A_1$.

We may write $Q = \mathcal{V}(x_1^2 - x_2x_3)$ where x_1, x_2, x_3 are linearly independent elements of A_1 and $x_ix_j - x_jx_i = \alpha_{ij}\Omega$ where $\alpha_{ij} \in k$ for all $i, j \in \{1, 2, 3\}$. The element $y_1 = \alpha_{23}x_1 - \alpha_{13}x_2 + \alpha_{12}x_3$ commutes in A with the elements of $kx_1 + kx_2 + kx_3$. The plane $\mathcal{V}(y_1) \subset \mathbb{P}^3$ meets Q in two lines (counted with multiplicity).

Suppose $\mathcal{V}(y_1)$ meets Q in two distinct lines $\mathcal{V}(y_1, y_2)$ and $\mathcal{V}(y_1, y_3)$. We may choose y_2 and y_3 such that $Q = \mathcal{V}(y_1^2 - y_2y_3)$. Hence, in this case, we may assume that there exist linearly independent elements $x_1, x_2, x_3 \in A_1$ such that $Q = \mathcal{V}(x_1^2 - x_2x_3)$ and $x_1x_2 = x_2x_1$ and $x_1x_3 = x_3x_1$ in A. In particular, we have that $x_1^2 - x_2x_3 \in k\Omega$ and $x_1^2 - x_3x_2 \in k\Omega$, since these are elements of A_2 which vanish on Q.

If $x_2x_3 = x_3x_2$ but $x_1^2 \neq x_2x_3$, then one may verify by computation that Ω is normal in A. However, if $x_2x_3 = x_3x_2$ and $x_1^2 = x_2x_3$, then A maps onto an algebra isomorphic to $k\langle x, y\rangle/\langle \beta x^2\rangle$ for some $\beta \in k$. In this case, A has infinite Gelfand-Kirillov dimension, which contradicts the regularity hypothesis on A.

If $x_2x_3 \neq x_3x_2$, then there exists $x_4 \in A_1\setminus$ span$\{x_1, x_2, x_3\}$ such that $x_2x_4 = x_4x_2$ and $x_3x_4 = x_4x_3$. If, in addition, $x_1^2 \neq x_2x_3$ and $x_1^2 \neq x_3x_2$, then it is straightforward to show that Ω is normal in A. So we may assume that $x_1^2 = x_2x_3$. If $x_1x_4 = x_4x_1$, then we have the algebra of Example 1.6, which is not regular. If $x_1x_4 \neq x_4x_1$, then Ω is central in A. Similar arguments apply if instead $x_1^2 = x_3x_2$.

It remains to consider the case where $\mathcal{V}(y_1)$ meets Q in a double line. This implies that there exist linearly independent elements $x_1, x_2, x_3 \in A_1$ such that $x_1x_2 = x_2x_1$ and $x_1x_3 = x_3x_1$ in A and $Q = \mathcal{V}(x_3^2 - x_1x_2)$. In particular, $x_3^2 - x_1x_2 \in k\Omega$ and $x_3^2 - x_2x_1 \in k\Omega$. If $x_2x_3 = x_3x_2$, then the first part of the above argument implies that either Ω is normal in A, or that A has infinite Gelfand-Kirillov dimension, which contradicts our hypothesis.

Hence we may assume that $x_2x_3 \neq x_3x_2$. It follows that there exists an element $x_4 \in A_1\setminus$ span$\{x_1, x_2, x_3\}$ such that $x_2x_4 = x_4x_2$ and $x_3x_4 = x_4x_3$. If $x_3^2 \neq x_1x_2$ and $x_3^2 \neq x_2x_1$, then it may be verified by computation that Ω is normal in A. Thus, we may assume that $x_3^2 = x_1x_2$. If $x_1x_4 = x_4x_1$, then A has infinite Gelfand-Kirillov dimension, thereby contradicting our hypothesis; but if $x_1x_4 \neq x_4x_1$, then Ω is central in A. Similar arguments apply if instead $x_3^2 = x_2x_1$. ∎

The following result demonstrates that a consequence of Lemma 1.7 is that R satisfies the good homological properties used in Theorem 1.4.

Corollary 1.8. *The algebra R is a noetherian domain, is Auslander-regular of global dimension four, and satisfies the Cohen-Macaulay property.*

Proof. The proof is identical to that for the rank four case in [12, Corollary 1.6]. ■

2. The Regular Algebras R whose Point Scheme is not the Quadric Q

Suppose $R \in \mathcal{R}(Q, \tau)$ where $R = T(V^*)/I$ as in Section 1, and identify $\mathbb{P}(V)$ with the copy of $\mathbb{P}^3$ which contains the quadric Q. By Theorem 1.4, the zero locus Γ_2 of the defining relations of R in $\mathbb{P}^3 \times \mathbb{P}^3$ is a subscheme of $\mathbb{P}^3 \times \mathbb{P}^3$ and is the graph of an automorphism σ of a subscheme P of $\mathbb{P}^3$. Moreover, $Q \subseteq P$ and $\sigma|_Q = \tau$. In this section we consider those algebras $R \in \mathcal{R}(Q, \tau)$ for which the point scheme P is not equal to Q.

The results in [12, §2.1] concerning regular algebras whose point scheme contains a rank four quadric apply to regular algebras whose point scheme contains a rank three quadric. In particular, the proofs of the following five results are analogous to their counterparts (Lemma 1.14 and Propositions 2.4 and 2.7 and Theorems 2.6 and 2.9) in [12].

Lemma 2.1. *If the point scheme P is not the quadric Q, then R is determined by the geometric data (P, σ); that is,*
$$R \cong \frac{T(V^*)}{\langle f \in V^* \otimes V^* \colon f|_{\Gamma_2} = 0 \rangle},$$
where Γ_2 is the zero locus of the defining relations of R in $\mathbb{P}(V) \times \mathbb{P}(V)$. ■

Proposition 2.2. *If the point scheme P is not the quadric Q, then there exists a normal element in R_1.* ■

Theorem 2.3. *If the point scheme P of R is not the quadric Q, then $\tau \in Aut(R)$. In this case, R is a twist by τ of a regular algebra R' which maps onto the (commutative) homogeneous coordinate ring $S(Q)$ of Q, such that $gldim(R') = 4$ and the Hilbert series of R' is $H_{R'}(t) = (1-t)^{-4}$. Moreover R' contains two linearly independent central elements of degree one.* ■

Proposition 2.4. *If the point scheme P of R is not the quadric Q, then either*

(a) $P = \mathbb{P}^3$, *or*

(b) $P = Q \cup L$ *where L is a line in $\mathbb{P}^3$ that meets Q in two points (counted with multiplicity), or*

(c) *the reduced scheme P_{red} associated to P is the quadric Q but the scheme P contains a double line L of (all) multiple points on P, where L corresponds to a line on Q (we denote this situation by $P = Q \uplus L$).*

In cases (b) and (c) there is a regular normalizing sequence $\{v_1, v_2\} \subset R_1$ *such that* $L = \mathcal{V}(v_1, v_2)$. ■

Theorem 2.5. *If the point scheme P of R is not the quadric Q, then R is a twist by τ of a regular central extension of a regular algebra of global dimension three.* ■

To classify the algebras R which have $P \neq Q$, we first classify those R for which $\tau =$ identity. Case (b) in Proposition 2.4 splits into three subcases depending on how the line L meets Q; in Proposition 2.6 those subcases are labelled (b), (b′) and (b″) respectively.

Proposition 2.6. *Suppose the point scheme P of R is not the quadric Q. If $\tau =$ identity, then there exist generators $x_1, \dots, x_4$ for R such that R has six quadratic defining relations given by one of the following cases. In each case, $Q = \mathcal{V}(x_1^2 - x_2x_3)$ and $\sigma|_Q = \tau =$ identity.*

(a) $R = S(\mathbb{P}^3) = k[x_1, \dots, x_4]$ *with defining relations* $x_ix_j = x_jx_i$ *for* $1 \leq i, j \leq 4$. *In this case* $(P, \sigma) = (\mathbb{P}^3,\ id)$.

(b) $R = k[x_1, \dots, x_4]$ *with defining relations*

$$\begin{aligned} x_1x_2 &= x_2x_1, & x_2x_4 &= x_4x_2, \\ x_1x_3 &= x_3x_1, & x_3x_4 &= x_4x_3, \\ x_1x_4 &= x_4x_1, & x_2x_3 - x_3x_2 &= \alpha(x_1^2 - x_2x_3), \end{aligned}$$

where $\alpha \in k^\times \setminus \{-1\}$. *In this case,* $P = Q \cup L$ *where* $L = \mathcal{V}(x_1, x_4)$ *(so L intersects Q at two distinct points) and* $\sigma|_L(0, x_2, x_3, 0) = (0, (\alpha+1)x_2, x_3, 0)$.

(b′) $R = k[x_1, \dots, x_4]$ *with defining relations*

$$\begin{aligned} x_3x_1 &= x_1x_3, & x_4x_1 &= x_1x_4, \\ x_3x_2 &= x_2x_3, & x_4x_2 &= x_2x_4, \\ x_3x_4 &= x_4x_3, & x_1x_2 - x_2x_1 &= x_1^2 - x_2x_3. \end{aligned}$$

In this case, $P = Q \cup L$ *where* $L = \mathcal{V}(x_3, x_4)$ *(so L is tangential to Q at a nonsingular point of Q) and* $\sigma|_L(x_1, x_2, 0, 0) = (x_1, x_1 + x_2, 0, 0)$.

(b″) $R = k[x_1, \dots, x_4]$ *with defining relations*

$$\begin{aligned} x_2x_1 &= x_1x_2, & x_3x_1 &= x_1x_3, \\ x_2x_3 &= x_3x_2, & x_3x_4 &= x_4x_3, \\ x_2x_4 &= x_4x_2, & x_1x_4 - x_4x_1 &= x_1^2 - x_2x_3. \end{aligned}$$

In this case, $P = Q \cup L$ *where* $L = \mathcal{V}(x_2, x_3)$ *(so L is tangential to Q at the singular point of Q) and* $\sigma|_L(x_1, 0, 0, x_4) = (x_1, 0, 0, x_1 + x_4)$.

(c) $R = k[x_1, \dots, x_4]$ *with defining relations*

$$\begin{aligned} x_1x_2 &= x_2x_1, & x_2x_3 &= x_3x_2, \\ x_1x_3 &= x_3x_1, & x_2x_4 &= x_4x_2, \\ x_1x_4 &= x_4x_1, & x_3x_4 - x_4x_3 &= x_1^2 - x_2x_3. \end{aligned}$$

In this case, $P = Q \uplus L$ *where* $L = \mathcal{V}(x_1, x_2) \subset Q$ *and* $\sigma \in Aut(P)$ *is uniquely determined by its action on the affine open sets* $P \setminus \mathcal{V}(x_i)$; *that is, its action on the corresponding localized rings* $S(P)[x_i^{-1}]$, *where* $S(P)$ *is the homogeneous coordinate* *ring of P, is as follows: on $S(P)[x_1^{-1}]$ and on $S(P)[x_2^{-1}]$, we have that σ is the identity, but on $S(P)[x_3^{-1}]$, we have*

$$\sigma(x_1, \dots, x_4) = (x_1, x_2, x_3, x_4 - x_3^{-1}(x_1^2 - x_2 x_3)),$$

and on $S(P)[x_4^{-1}]$, we have

$$\sigma(x_1, \dots, x_4) = (x_1, x_2, x_3 + x_4^{-1}(x_1^2 - x_2 x_3), x_4).$$

All such algebras are regular with Hilbert series $H(t) = (1-t)^{-4}$ and have point scheme $P \neq Q$.

Proof. Under our hypotheses R maps onto $S(Q)$. Hence, if $x_1, \dots, x_4$ are generators for $S(Q)$ such that $Q = \mathcal{V}(x_1^2 - x_2 x_3)$, then the defining relations of R are nonzero linear combinations of $x_1^2 - x_2 x_3$ and $x_i x_j - x_j x_i$ for all i, j.

(a) If $P = \mathbb{P}^3$, then $\sigma \in \operatorname{Aut}(\mathbb{P}^3)$ so is linear. Hence $\sigma = \tau =$ identity. Combining this with the preceding observation yields that $R = S(\mathbb{P}^3)$.

(b) If L is a line in $\mathbb{P}^3$ which meets Q in two distinct points, then there exists a choice of coordinates $x_1, \dots, x_4$ such that $L = \mathcal{V}(x_1, x_4)$ and $Q = \mathcal{V}(x_1^2 - x_2 x_3)$. By Proposition 2.4, it follows that x_1 and x_4 are central elements in R. Thus, five of the six defining relations of R are determined. However, by Theorem 2.5, R/Rx_1 is a domain, so $x_1^2 - x_2 x_3$ is nonzero in R; moreover, $x_2 x_3 - x_3 x_2$ is also nonzero since $P \neq \mathbb{P}^3$. It follows that $x_2 x_3 - x_3 x_2 = \alpha(x_1^2 - x_2 x_3) \in k^\times \Omega$ for some $\alpha \in k^\times$, and this gives the sixth defining relation. It remains to verify which algebras with such defining relations are regular. If $\alpha = -1$, then such an algebra is isomorphic to the algebra in Example 1.6, which is not regular. However, if $\alpha \in k^\times \setminus \{-1\}$, then the algebra is an iterated Ore extension $k[x_1, x_2, x_4][x_3; \nu, \delta]$ where $\nu(x_1) = x_1$, $\nu(x_4) = x_4$, $\nu(x_2) = (\alpha+1)x_2$, $\delta(x_1) = 0 = \delta(x_4)$ and $\delta(x_2) = -\alpha x_1^2$, and it may be verified that δ is a ν-derivation. It follows that such an algebra is regular of global dimension four with Hilbert series $H(t) = (1-t)^{-4}$.

The automorphism $\sigma|_L$ may be found by factoring out the ideal $Rx_1 + Rx_4$.

(b′) If L is a line in $\mathbb{P}^3$ which meets Q tangentially at a nonsingular point of Q, then there exists a choice of coordinates $x_1, \dots, x_4$ such that $L = \mathcal{V}(x_3, x_4)$, and $Q = \mathcal{V}(x_1^2 - x_2 x_3)$. By Proposition 2.4, it follows that x_3 and x_4 are central elements in R. As in part (b), one may show that $x_1 x_2 - x_2 x_1 = \alpha(x_1^2 - x_2 x_3) \in k^\times \Omega$ for some $\alpha \in k^\times$, which yields the last defining relation. Mapping $x_2 \mapsto \alpha x_2$, $x_3 \mapsto \alpha^{-1} x_3$, $x_1 \mapsto x_1$ and $x_4 \mapsto x_4$ gives the defining relations in (b′). This algebra is an iterated Ore extension $k[x_1, x_3, x_4][x_2; \mu, \delta]$ where $\mu(x_3) = x_3$, $\mu(x_4) = x_4$, $\mu(x_1) = x_1 + x_3$, $\delta(x_3) = 0 = \delta(x_4)$, $\delta(x_1) = -x_1^2$, and it may be verified that δ is a μ-derivation.

The automorphism $\sigma|_L$ may be found by factoring out the ideal $Rx_3 + Rx_4$.

(b″) If L is a line in $\mathbb{P}^3$ which meets Q tangentially at the singular point of Q, then there exists a choice of coordinates $x_1, \dots, x_4$ such that $L = \mathcal{V}(x_2, x_3)$, and $Q = \mathcal{V}(x_1^2 - x_2 x_3)$. By

Proposition 2.4, it follows that x_2 and x_3 are central elements in R. An argument similar to that in (b) yields that $x_1x_4 - x_4x_1 = \alpha(x_1^2 - x_2x_3) \in k^\times\Omega$ for some $\alpha \in k^\times$. However, all such algebras are isomorphic (by rechoosing x_4), so we may take $\alpha = 1$. This algebra is an iterated Ore extension $k[x_1, x_2, x_3][x_4; \mathrm{id}, \delta]$ where $\delta(x_2) = 0 = \delta(x_3)$ and $\delta(x_1) = x_2x_3 - x_1^2$.

The automorphism $\sigma|_L$ may be found by factoring out the ideal $Rx_2 + Rx_3$.

(c) If L is a line on Q, then there exists a choice of coordinates $x_1, \dots, x_4$ such that $L = \mathcal{V}(x_1, x_2)$, and $Q = \mathcal{V}(x_1^2 - x_2x_3)$, so that x_1 and x_2 are central elements in R as above. An argument similar to that in (b) yields that $x_3x_4 - x_4x_3 = \alpha(x_1^2 - x_2x_3) \in k^\times\Omega$ for some $\alpha \in k^\times$. As in (b$''$), all such algebras are isomorphic, so we may take $\alpha = 1$. This algebra is an iterated Ore extension $k[x_1, x_2, x_3][x_4; \mathrm{id}, \delta]$ where $\delta(x_1) = 0 = \delta(x_2)$ and $\delta(x_3) = x_2x_3 - x_1^2$.

To compute the action of the automorphism, set $Y_i' = x_i \otimes 1$ and $Z_i' = 1 \otimes x_i$ for all $i = 1, \dots, 4$ in the defining relations of the algebra. This yields that $S(P)$ is isomorphic to the commutative algebra $k[Y_1, \dots, Y_4, Z_1, \dots, Z_4]$ with defining relations given by

$$\begin{array}{ll} Y_1Z_2 = Y_2Z_1, & Y_2Z_3 = Y_3Z_2, \\ Y_1Z_3 = Y_3Z_1, & Y_2Z_4 = Y_4Z_2, \\ Y_1Z_4 = Y_4Z_1, & Y_3Z_4 - Y_4Z_3 = Y_1Z_1 - Y_2Z_3. \end{array}$$

Since the point scheme is the graph of an automorphism, the subalgebra generated by the Y_i (respectively, the Z_i) is the homogeneous coordinate ring of the left (respectively, right) projection of P into $\mathbb{P}^3$ and is isomorphic to $S(P)$. For each $S(P)[x_j^{-1}]$ (that is, on each affine open set $\mathbb{P}^3 \setminus \mathcal{V}(x_j)$), set $Y_i = Y_i'/Y_j'$ and $Z_i = Z_i'/Z_j'$ for all $i \neq j$. By definition of the automorphism associated with a point scheme, it follows that, on each of these localizations, σ is given by $\sigma(Z_i) = Y_i$ for all i. Hence, computing σ entails rewriting the Y_i in terms of the Z_i in each $S(P)[x_j^{-1}]$, and rewriting in terms of the x_i.

For $S(P)[x_j^{-1}]$ where $j = 1, 2$, σ is the identity. For $S(P)[x_3^{-1}]$, we have $Y_1Z_4 = Y_4Z_1$, $Y_2Z_4 = Y_4Z_2$, $Y_1 = Z_1$, $Y_2 = Z_2$ and $Y_4 - Z_4 = Z_1^2 - Z_2Z_3$, so that $Y_4 = Z_4 - (Z_1^2 - Z_2Z_3)$. For $S(P)[x_4^{-1}]$, we have $Y_1Z_3 = Y_3Z_1$, $Y_2Z_3 = Y_3Z_2$, $Y_1 = Z_1$, $Y_2 = Z_2$ and $Y_3 - Z_3 = Z_1^2 - Z_2Z_3$, so that $Y_3 = Z_3 + (Z_1^2 - Z_2Z_3)$. The result follows. ■

Corollary 2.7. *We have $\tau \in Aut(P)$ and τ commutes with σ on P.*

Proof. The proof is analogous to that for [12, Corollary 2.13]. ■

Recall that x^τ denotes $\tau(x)$ for all $x \in R$.

Theorem 2.8. *The regular algebra R is isomorphic to one of the following algebras if and only if the point scheme P of R is not the quadric Q.*

(a) *$R = k[x_1, \dots, x_4]$ with defining relations $x_i^\tau x_j = x_j^\tau x_i$ for $1 \le i, j \le 4$ for all $\tau \in Aut(\mathbb{P}^3)$. In this case, R is the twist of the polynomial ring $S(\mathbb{P}^3)$ by τ, so $(P, \sigma) = (\mathbb{P}^3, \tau)$.*

(b) $R = k[x_1, \dots, x_4]$ *with defining relations*

$$\begin{aligned} x_1^\tau x_2 &= x_2^\tau x_1, & x_2^\tau x_4 &= x_4^\tau x_2,\\ x_1^\tau x_3 &= x_3^\tau x_1, & x_3^\tau x_4 &= x_4^\tau x_3,\\ x_1^\tau x_4 &= x_4^\tau x_1, & x_2^\tau x_3 - x_3^\tau x_2 &= \alpha(x_1^\tau x_1 - x_2^\tau x_3), \end{aligned}$$

where $\alpha \in k^\times \setminus \{-1\}$*, where* τ *is given (with respect to a basis dual to the* x_i*) by*

$$\tau \in k^\times \begin{pmatrix} ab & 0 & 0 & 0\\ 0 & a^2 & 0 & 0\\ 0 & 0 & b^2 & 0\\ c & 0 & 0 & 1 \end{pmatrix} \tag{$*$}$$

where $a, b \in k^\times$*,* $c \in k$*; if* $\alpha = -2$*, then* τ *may be given by* $(*)$ *or*

$$\tau \in k^\times \begin{pmatrix} ab & 0 & 0 & 0\\ 0 & 0 & b^2 & 0\\ 0 & a^2 & 0 & 0\\ c & 0 & 0 & 1 \end{pmatrix}$$

where $a, b \in k^\times$*,* $c \in k$*. In this case,* $P = Q \cup L$ *where* $L = \mathcal{V}(x_1, x_4)$ *(so* L *intersects* Q *at two distinct points),* $\sigma|_Q = \tau$*, and* $\sigma|_L(0, x_2, x_3, 0) = \tau(0, (\alpha+1)x_2, x_3, 0)$*.*

(b′) $R = k[x_1, \dots, x_4]$ *with defining relations*

$$\begin{aligned} x_3^\tau x_1 &= x_1^\tau x_3, & x_4^\tau x_1 &= x_1^\tau x_4,\\ x_3^\tau x_2 &= x_2^\tau x_3, & x_4^\tau x_2 &= x_2^\tau x_4,\\ x_3^\tau x_4 &= x_4^\tau x_3, & x_1^\tau x_2 - x_2^\tau x_1 &= x_1^\tau x_1 - x_2^\tau x_3, \end{aligned}$$

where τ *is given (with respect to a basis dual to the* x_i*) by*

$$\tau \in k^\times \begin{pmatrix} a^2 & 0 & ab & 0\\ 2ab & a^2 & b^2 & 0\\ 0 & 0 & a^2 & 0\\ 0 & 0 & c & 1 \end{pmatrix}$$

where $a \in k^\times$*,* $b, c \in k$*. In this case,* $P = Q \cup L$ *where* $L = \mathcal{V}(x_3, x_4)$ *(so* L *is tangential to* Q *at a nonsingular point of* Q*),* $\sigma|_Q = \tau$*, and* $\sigma|_L(x_1, x_2, 0, 0) = \tau(x_1, x_1 + x_2, 0, 0)$*.*

(b″) $R = k[x_1, \dots, x_4]$ *with defining relations*

$$\begin{aligned} x_2^\tau x_1 &= x_1^\tau x_2, & x_3^\tau x_1 &= x_1^\tau x_3,\\ x_2^\tau x_3 &= x_3^\tau x_2, & x_3^\tau x_4 &= x_4^\tau x_3,\\ x_2^\tau x_4 &= x_4^\tau x_2, & x_1^\tau x_4 - x_4^\tau x_1 &= x_1^\tau x_1 - x_2^\tau x_3, \end{aligned}$$

where τ *is given (with respect to a basis dual to the* x_i*) by*

$$\tau \in k^\times \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & a & 0 & 0\\ 0 & 0 & a^{-1} & 0\\ b & c & d & 1 \end{pmatrix} \qquad \text{or} \qquad \tau \in k^\times \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 0 & a^{-1} & 0\\ 0 & a & 0 & 0\\ b & c & d & 1 \end{pmatrix}$$

where $a \in k^\times$*,* $b, c, d \in k$*. In this case,* $P = Q \cup L$ *where* $L = \mathcal{V}(x_2, x_3)$ *(so* L *is tangential to* Q *at the singular point of* Q*),* $\sigma|_Q = \tau$*, and* $\sigma|_L(x_1, 0, 0, x_4) = \tau(x_1, 0, 0, x_1 + x_4)$*.*

(c) $R = k[x_1, \dots, x_4]$ *with defining relations*

$$\begin{array}{cc} x_1^\tau x_2 = x_2^\tau x_1, & x_2^\tau x_3 = x_3^\tau x_2, \\ x_1^\tau x_3 = x_3^\tau x_1, & x_2^\tau x_4 = x_4^\tau x_2, \\ x_1^\tau x_4 = x_4^\tau x_1, & x_3^\tau x_4 - x_4^\tau x_3 = x_1^\tau x_1 - x_2^\tau x_3, \end{array}$$

where τ is given (with respect to a basis dual to the x_i) by

$$\tau \in k^\times \begin{pmatrix} a & b & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2ab & b^2 & a^2 & 0 \\ c & d & e & 1 \end{pmatrix}$$

where $a \in k^\times$, $b, c, d, e \in k$. In this case, $P = Q \uplus L$ where $L = \mathcal{V}(x_1, x_2) \subset Q$, $\sigma|_Q = \tau$, and $\sigma \in Aut(P)$ is uniquely determined by its action on the affine open sets $P \setminus \mathcal{V}(x_i)$; that is, its action on the corresponding localized rings $S(P)[x_i^{-1}]$, where $S(P)$ is the homogeneous coordinate ring of P, is as follows: on $S(P)[x_1^{-1}]$ and on $S(P)[x_2^{-1}]$, we have that $\sigma = \tau^{-1}$, but on $S(P)[x_3^{-1}]$, we have

$$\sigma(x_1, \dots, x_4) = \tau^{-1}(x_1, x_2, x_3, x_4 - x_3^{-1}(x_1^2 - x_2 x_3)),$$

and on $S(P)[x_4^{-1}]$, we have

$$\sigma(x_1, \dots, x_4) = \tau^{-1}(x_1, x_2, x_3 + x_4^{-1}(x_1^2 - x_2 x_3), x_4).$$

All such algebras are regular with Hilbert series $H(t) = (1-t)^{-4}$ and have point scheme $P \neq Q$.

Proof. By Theorem 2.3 and Proposition 2.6, it follows that $P \neq Q$ if and only if R is a twist by τ of one of the algebras given in Proposition 2.6. Hence it suffices to classify the graded degree zero automorphisms of the algebras listed in Proposition 2.6 which are determined by (P, σ) where $\sigma|_Q =$ identity. Applying Corollary 2.7 to (a), (b), (b′) and (b″) yields that the matrix for τ is determined by the conditions: $\tau \in \mathrm{Aut}(Q) \cap \mathrm{Aut}(L)$ and $\tau|_L \circ \sigma|_L = \sigma|_L \circ \tau|_L$ (since $\sigma|_Q =$ identity). A computation completes the proof for (a), (b), (b′) and (b″).

For (c), Corollary 2.7 implies that $\tau \in \mathrm{Aut}(Q) \cap \mathrm{Aut}(L)$, but it is simpler to apply τ to the defining relations of the algebra in Proposition 2.6 than to work with the commutativity of τ and σ. Any element of $\mathrm{Aut}(Q) \cap \mathrm{Aut}(L)$ may be given (with respect to a basis dual to the x_i) by

$$\tau \in k^\times \begin{pmatrix} a & \mu^2 b & 0 & 0 \\ 0 & \mu^2 & 0 & 0 \\ 2ab & \mu^2 b^2 & a^2\mu^{-2} & 0 \\ c & d & e & 1 \end{pmatrix}, \tag{$\dagger$}$$

where $a, \mu \in k^\times$, $b, c, d, e \in k$. The matrix given in (†) maps the span of the defining relations of the algebra back to itself if and only if $\mu^2 = 1$, so the result follows. ■

Proposition 2.9. *If $P = \mathbb{P}^3$, then the isomorphism classes of left (respectively, right) line modules over R are in one-to-one correspondence with the lines in $\mathbb{P}^3$. If $P = Q \cup L$ or if*

$P = Q \uplus L$, then the isomorphism classes of left (respectively, right) line modules over R are in one-to-one correspondence with the lines in $\mathbb{P}^3$ that either lie on Q or intersect L.

Proof. The proof is identical to that for [12, Proposition 2.15]. ■

3. The Regular Algebras R whose Point Scheme is the Quadric Q

We continue to assume $R \in \mathcal{R}(Q, \tau)$ with point scheme P. The results of this section address the cases where $P = Q$.

Lemma 3.1. *The following are equivalent:*

(a) *the point scheme P of R is the quadric Q*
(b) *there are no normal elements in R_1*
(c) *the map $\tau \notin Aut(R)$.*

Proof. The proof that (a) implies (b) and that (a) implies (c) is identical to those given in [12, §4.1]. Moreover, (b) implies (a) is Proposition 2.2, and (c) implies (a) is Theorem 2.3. ■

Proposition 3.2. *If the point scheme P of R is the quadric Q, then there exist generators $x_1, \ldots, x_4$ for R such that $Q = \mathcal{V}(x_1^2 - x_2x_3)$ or $Q = \mathcal{V}(x_3^2 - x_1x_2)$ where R has defining relations*

$$\begin{gathered} x_1^\tau x_2 = x_2^\tau x_1, \qquad\qquad x_2^\tau x_4 = x_4^\tau x_2, \\ x_1^\tau x_3 = x_3^\tau x_1, \qquad\qquad x_3^\tau x_4 = x_4^\tau x_3, \\ x_1^\tau x_4 - x_4^\tau x_1 = x_2^\tau x_3 - x_3^\tau x_2, \end{gathered}$$

$$x_2^\tau x_3 - x_3^\tau x_2 = \alpha(x_1^\tau x_1 - x_2^\tau x_3) \qquad \text{if } \ Q = \mathcal{V}(x_1^2 - x_2x_3), \tag{1}$$

$$\text{or} \quad x_2^\tau x_3 - x_3^\tau x_2 = \alpha(x_3^\tau x_3 - x_1^\tau x_2) \qquad \text{if } \ Q = \mathcal{V}(x_3^2 - x_1x_2), \tag{2}$$

for some $\alpha \in k^\times$, where $\alpha \neq -1$ in (1). Moreover we may take $\Omega = x_2^\tau x_3 - x_3^\tau x_2$.

Proof. As in the proof of Lemma 1.7, there exist linearly independent elements $x_1, \ldots, x_4 \in R_1$ such that $x_1^\tau x_2 = x_2^\tau x_1$, $x_1^\tau x_3 = x_3^\tau x_1$, and either (1) $Q = \mathcal{V}(x_1^2 - x_2x_3)$ with $x_1^\tau x_1 - x_2^\tau x_3 \in k\Omega$ and $x_1^\tau x_1 - x_3^\tau x_2 \in k\Omega$, or (2) $Q = \mathcal{V}(x_3^2 - x_1x_2)$ with $x_3^\tau x_3 - x_1^\tau x_2 \in k\Omega$ and $x_3^\tau x_3 - x_2^\tau x_1 \in k\Omega$. However, by Corollary 1.8, R is noetherian and Auslander-regular, so by [8, Proposition 2.8] we have $x_1^\tau x_1 - x_2^\tau x_3 \in k^\times(x_1^\tau x_1 - x_3^\tau x_2) \in k^\times\Omega$ in (1) and $x_3^\tau x_3 - x_1^\tau x_2 \in k^\times(x_3^\tau x_3 - x_2^\tau x_1) \in k^\times\Omega$ in (2).

If $x_2^\tau x_3 = x_3^\tau x_2$, then cases (1) and (2) are equivalent, so we focus on (1). In this situation $x_i^\tau x_4 - x_4^\tau x_i = \alpha_i(x_1^\tau x_1 - x_2^\tau x_3)$ where $\alpha_i \in k$. Since Q is invariant under τ and is the point scheme of R, it is also the point scheme of the algebra $R' = k[y_1, y_2, y_3, y_4]$ which has defining relations: $y_iy_j = y_jy_i$ for all $i, j \in \{1, 2, 3\}$ and $y_iy_4 - y_4y_i = \alpha_i(y_1^2 - y_2y_3)$ where $i \in \{1, 2, 3\}$ and α_i as above. The algebra R' is an iterated Ore extension; it is $S(\mathbb{P}^3)$ if $\alpha_i = 0$ for all i, otherwise it contains central elements of degree one, namely $\alpha_ix_j - \alpha_jx_i$ for all i, j. In either case, R' maps onto the homogeneous coordinate ring $S(Q)$ of a rank three quadric in $\mathbb{P}^3$. Thus, by Lemma 3.1, R' is isomorphic to one of the algebras given in Theorem 2.8, and so the point scheme is not the quadric Q, which contradicts our hypothesis.

Hence we may assume that $x_2^\tau x_3 - x_3^\tau x_2 \in k^\times \Omega$. It follows that we may rechoose $x_4 \in R_1 \setminus \operatorname{span}\{x_1, x_2, x_3\}$ such that $x_2^\tau x_4 = x_4^\tau x_2$ and $x_3^\tau x_4 = x_4^\tau x_3$. Suppose $x_1^\tau x_4 = x_4^\tau x_1$, then, since Q is invariant under τ and is the point scheme of R, it is also the point scheme of the algebra $R'' = k[z_1, \ldots, z_4]$ with defining relations given by z_1 and z_4 are central, and in case (1) $z_2z_3 - z_3z_2 = \alpha(z_1^2 - z_2z_3)$ whereas in case (2) we have $z_2z_3 - z_3z_2 = \alpha(z_3^2 - z_1z_2)$ for some $\alpha \in k^\times$. A straightforward computation shows that the point scheme of R'' is $\mathcal{V}(z_1^2 - z_2z_3) \cup \mathcal{V}(z_1, z_4)$ in case (1) and $\mathcal{V}(z_3^2 - z_1z_2) \cup \mathcal{V}(z_1, z_4)$ in case (2), neither of which is Q. Hence, $x_1^\tau x_4 \neq x_4^\tau x_1$ and so $x_1^\tau x_4 - x_4^\tau x_1 \in k^\times \Omega$.

By rescaling x_4, we obtain that the defining relations of R have the required form. Moreover, in (1) we have that $\alpha \neq -1$, since otherwise there exists a nonunique choice of $a, b \in R_1$ such that $ax_2 = bx_1$, which is false by [8, Proposition 2.8] and Corollary 1.8. ■

The next two results are proved in [12] where $Q \subset \mathbb{P}^3$ has rank four, but the proofs are identical to those for a rank three quadric $Q \subset \mathbb{P}^3$.

Lemma 3.3. *We have $R^! \cong S^!/\langle \omega \rangle$ where $\omega \in S_2^!$ vanishes on the defining relations of R, and $\omega \cdot \Omega = 1$. The element ω is normal and 1-regular in $S^!$.* ■

Lemma 3.4.

(a) *If $\psi \in Aut(S^!)$, then $\psi \circ \tau \in k\tau \circ \psi$.*

(b) *If $\psi \in Aut(S^!)$ and if $S = A/\langle \Omega \rangle$ where A is any quadratic algebra and $\Omega \in A_2$, then $\psi \in Aut(A)$ if and only if $\omega^\psi \in k^\times \omega$ where $A^! = S^!/\langle \omega \rangle$.* ■

Let $x_1, \ldots, x_4$ be the generators of R from Proposition 3.2 and let $\eta_1, \ldots, \eta_4$ be a dual basis to $x_1, \ldots, x_4$, and let the normal element $\Omega = x_2^\tau x_3 - x_3^\tau x_2$. In this notation, the defining relations of S are

$$\begin{array}{llll} x_1^\tau x_2 = x_2^\tau x_1, & x_1^\tau x_4 = x_4^\tau x_1, & x_2^\tau x_4 = x_4^\tau x_2, & x_i^\tau x_i = x_2^\tau x_j, \\ x_1^\tau x_3 = x_3^\tau x_1, & x_2^\tau x_3 = x_3^\tau x_2, & x_3^\tau x_4 = x_4^\tau x_3, & \end{array}$$

where $Q = \mathcal{V}(x_i^2 - x_2x_j)$ and $\{i, j\} = \{1, 3\}$. The algebra $S^!$ has Hilbert series $H(t) = (1+t)^3(1-t)^{-1}$ and defining relations

$$\begin{array}{lll} \eta_i^{\tau^{-1}}\eta_i = 0, & \eta_i^{\tau^{-1}}\eta_2 = -\eta_2^{\tau^{-1}}\eta_i, & \eta_4^{\tau^{-1}}\eta_2 = -\eta_2^{\tau^{-1}}\eta_4, \\ \eta_2^{\tau^{-1}}\eta_2 = 0, & \eta_3^{\tau^{-1}}\eta_1 = -\eta_1^{\tau^{-1}}\eta_3, & \eta_4^{\tau^{-1}}\eta_3 = -\eta_3^{\tau^{-1}}\eta_4, \\ \eta_4^{\tau^{-1}}\eta_4 = 0, & \eta_4^{\tau^{-1}}\eta_1 = -\eta_1^{\tau^{-1}}\eta_4, & \eta_i^{\tau^{-1}}\eta_i = -(\eta_j^{\tau^{-1}}\eta_2 + \eta_2^{\tau^{-1}}\eta_j), \end{array}$$

where $\{i, j\} = \{1, 3\}$ as in the defining relations for S. Clearly, $\tau \in \operatorname{Aut}(S^!)$ since $\tau \in \operatorname{Aut}(S)$. With these defining relations and those in Proposition 3.2, it follows that

$$\omega = \begin{cases} \eta_1^{\tau^{-1}}\eta_4 + \eta_2^{\tau^{-1}}\eta_3 + (1+\alpha^{-1})\eta_1^{\tau^{-1}}\eta_1 & \text{if } \ Q = \mathcal{V}(x_1^2 - x_2x_3) \\ \eta_1^{\tau^{-1}}\eta_4 + \eta_2^{\tau^{-1}}\eta_3 + \alpha^{-1}\eta_3^{\tau^{-1}}\eta_3 & \text{if } \ Q = \mathcal{V}(x_3^2 - x_1x_2), \end{cases}$$

where $\alpha \in k^{\times}$ is defined in Proposition 3.2.

By Lemma 3.3, the element ω determines a map $\theta \in \mathrm{Aut}(S^!)$ by $\omega a = a^{\theta}\omega$ for all $a \in S^!$. Since $\theta \in \mathrm{Aut}(S^!)$ and $\omega^{\theta} = \omega$, we have $\theta \circ \tau \in k\tau \circ \theta$ and $\theta \in \mathrm{Aut}(R)$ by Lemma 3.4.

Lemma 3.5. *If the point scheme P of R is the quadric Q, then there exists a choice of generators $x_1, \ldots, x_4$ for R such that $Q = \mathcal{V}(x_1^2 - x_2x_3)$ and R has defining relations*

$$\begin{array}{lll} x_1^{\tau}x_2 = x_2^{\tau}x_1, & x_2^{\tau}x_4 = x_4^{\tau}x_2, & x_1^{\tau}x_4 - x_4^{\tau}x_1 = x_2^{\tau}x_3 - x_3^{\tau}x_2, \\ x_1^{\tau}x_3 = x_3^{\tau}x_1, & x_3^{\tau}x_4 = x_4^{\tau}x_3, & x_2^{\tau}x_3 - x_3^{\tau}x_2 = \alpha(x_1^{\tau}x_1 - x_2^{\tau}x_3) \end{array}$$

for some $\alpha \in k$ such that $\alpha(\alpha+1) \neq 0$. Moreover we may take $\Omega = x_2^{\tau}x_3 - x_3^{\tau}x_2$.

Proof. Suppose that R and $S^!$ have the defining relations given above, and that $Q = \mathcal{V}(x_3^2 - x_1x_2)$. By Proposition 3.2, Lemma 3.3 and the preceding remarks, the result would follow if $\theta \notin \mathrm{Aut}(S^!)$.

Since $\tau \in \mathrm{Aut}(S^!)$, we have that $\eta_i\eta_i^{\tau} = 0$ where $i \in \{1,2,4\}$. Thus, $\eta_i^{\theta}\omega\eta_i^{\tau} = \omega\eta_i\eta_i^{\tau} = 0$ where $i \in \{1,2,4\}$. Computing $\omega\eta_i^{\tau}$ and finding which elements annihilate it on the left shows that $\eta_1^{\theta} = \mu(\eta_3 + \alpha\eta_2 + \alpha^{-1}\eta_1)^{\tau^{-2}}$, $\eta_2^{\theta} = \nu k(\eta_2 + \alpha\eta_4)^{\tau^{-2}}$ and $\eta_4^{\theta} = \lambda\eta_4^{\tau^{-2}}$, where $\mu, \nu, \lambda \in k^{\times}$. Hence, for each $i \in \{1,2,4\}$, one may compute $\eta_i^{\theta}\omega$ which equals $\omega\eta_i$, and thus obtain three equal expressions for ω:

$$\begin{aligned} \omega &= \lambda(\eta_2^{\tau^{-2}}\eta_3^{\tau^{-1}} + \alpha^{-1}\eta_3^{\tau^{-2}}\eta_3^{\tau^{-1}} + a\eta_4^{\tau^{-1}}), \\ \omega &= \nu(\alpha\eta_3^{\tau^{-2}}\eta_4^{\tau^{-1}} + \alpha^{-1}\eta_3^{\tau^{-2}}\eta_3^{\tau^{-1}} + \eta_1^{\tau^{-2}}\eta_4^{\tau^{-1}} + b\eta_2^{\tau^{-1}}), \\ \omega &= \mu(-\eta_3^{\tau^{-2}}\eta_4^{\tau^{-1}} - \alpha\eta_2^{\tau^{-2}}\eta_4^{\tau^{-1}} + \alpha^{-2}\eta_3^{\tau^{-2}}\eta_3^{\tau^{-1}} + \alpha^{-1}\eta_2^{\tau^{-2}}\eta_3^{\tau^{-1}} + c\eta_1^{\tau^{-1}}), \end{aligned}$$

where $a, b, c \in (S^!)_1$. By equating coefficients, we find that $\mu = \nu = \lambda = 0$, which is a contradiction. ■

Lemma 3.6. *If $P = Q$, then R has defining relations as given in Lemma 3.5 where $\alpha = -2$. In this case, $\theta = \phi \circ \tau^{-2}$ where*

$$\phi = \lambda \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(with respect to the η_i) for some $\lambda \in k^{\times}$, and

$$\begin{aligned} \omega &= \eta_1^{\tau^{-1}}\eta_4 + \eta_2^{\tau^{-1}}\eta_3 + \tfrac{1}{2}\eta_1^{\tau^{-1}}\eta_1 \\ &= \lambda(-\eta_1^{\tau^{-2}}\eta_4^{\tau^{-1}} + \eta_2^{\tau^{-2}}\eta_3^{\tau^{-1}} + \tfrac{1}{2}\eta_1^{\tau^{-2}}\eta_1^{\tau^{-1}}). \end{aligned}$$

Moreover, $\tau^2 \in Aut(R)$, and ϕ and τ commute on $S^!$.

Proof. As in the proof of Lemma 3.5, we have that $\eta_i\eta_i^{\tau} = 0$ where $i \in \{2,3,4\}$, so that $\eta_i^{\theta}\omega\eta_i^{\tau} = \omega\eta_i\eta_i^{\tau} = 0$ where $i \in \{2,3,4\}$. Computing $\omega\eta_i^{\tau}$ and finding which elements annihilate it on the left shows that $\eta_i^\theta = \mu_i \eta_i^{\tau^{-2}}$, where $i \in \{2,3,4\}$ and the $\mu_i \in k^\times$. As in the proof of Lemma 3.5, one obtains three equal expressions for ω:

$$\begin{aligned}
\omega &= \mu_2(\eta_1^{\tau^{-2}}\eta_4^{\tau^{-1}} + (1+\alpha^{-1})\eta_1^{\tau^{-2}}\eta_1^{\tau^{-1}} + a\eta_2^{\tau^{-1}}),\\
\omega &= \mu_3(\eta_1^{\tau^{-2}}\eta_4^{\tau^{-1}} + \alpha^{-1}\eta_1^{\tau^{-2}}\eta_1^{\tau^{-1}} + b\eta_3^{\tau^{-1}}),\\
\omega &= \mu_4(\eta_2^{\tau^{-2}}\eta_3^{\tau^{-1}} + (1+\alpha^{-1})\eta_1^{\tau^{-2}}\eta_1^{\tau^{-1}} + c\eta_4^{\tau^{-1}}),
\end{aligned}$$

where $a, b, c \in (S^!)_1$. By equating coefficients, we find that $\mu_2 = \mu_3 = \pm\mu_4$. However, if the upper sign is used, then $\omega^\tau = \omega$ so that $\tau \in \mathrm{Aut}(R)$, which implies that $P \neq Q$ by Lemma 3.1, thereby contradicting the hypothesis. Using the lower sign, we find that $\alpha = -2$, and we obtain the claimed expressions for ω. Computing $\omega\eta_1$, which equals $\eta_1^\theta\omega$, it follows that $\eta_1^\theta = \mu_4\eta_1^{\tau^{-2}}$. Hence $\theta = \phi \circ \tau^{-2}$ where ϕ is given above (with $\lambda = \mu_4$). Since $\phi \in \mathrm{Aut}(S^!)$, we have that τ and ϕ commute on Q and hence on $\mathbb{P}^3$ by [12, Lemma 2.12]. It follows (by computation) that $\omega^\phi \in k^\times\omega$, whence $\phi \in \mathrm{Aut}(R)$, and, thus, so is τ^2. By the argument in [12, Lemma 4.5], one has $\phi \circ \tau = \pm\tau \circ \phi$ on $S^!$, which completes the proof since η_4 is an eigenvector of both ϕ and τ. ■

Lemma 3.7. *If $P = Q$, then (with respect to the η_i) the matrix representing τ has the form:*

$$(i)\ \tau \in k^\times \begin{pmatrix} -1 & 0 & 0 & 0\\ 0 & c & 0 & 0\\ 0 & 0 & 1/c & 0\\ b & 0 & 0 & 1 \end{pmatrix} \quad \text{or} \quad (ii)\ \tau \in k^\times \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 0 & c & 0\\ 0 & 1/c & 0 & 0\\ b & 0 & 0 & 1 \end{pmatrix},$$

for some $b \in k$, $c \in k^\times$.

Proof. By Lemma 3.6, τ and ϕ commute on $S^!$, and since $\tau \in \mathrm{Aut}(Q)$, it follows that

$$\tau \in k^\times \begin{pmatrix} a & 0 & 0 & 0\\ 0 & c & 0 & 0\\ 0 & 0 & d & 0\\ b & 0 & 0 & 1 \end{pmatrix} \quad \text{or} \quad \tau \in k^\times \begin{pmatrix} a & 0 & 0 & 0\\ 0 & 0 & c & 0\\ 0 & d & 0 & 0\\ b & 0 & 0 & 1 \end{pmatrix},$$

for some $a, c, d \in k^\times$, $b \in k$. Verifying that $\omega^\tau \in k^\times\omega$ yields the result. ■

We now change our viewpoint. Fix $\tau \in \mathrm{Aut}(Q)$ such that it has a matrix representative of the form given in Lemma 3.7 (with respect to the η_i). Let $A(\tau)$ denote a k-algebra on $x_1, \ldots, x_4$ with defining relations given by Lemma 3.5 where $\alpha = -2$. According to the above results, if $P = Q$, then the regular algebra R is of the form $A(\tau)$; but conceivably not every $A(\tau)$ is a regular algebra. We prove below that every $A(\tau)$ twists by a twisting system to $A(t)$ for a particular $t \in \mathrm{Aut}(Q)$. Since regularity is twisting system invariant, it then follows that either every $A(\tau)$ is regular, or none are regular. We refer the reader to [15] for details on twisting systems, and to [12, §4] for their application in the following results. We define ω (as above) to be the element of $(S^!)_2$ with the property that it vanishes on the defining relations of $A(\tau)$ and that $\omega \cdot \Omega = 1$ where $\Omega = x_2^\tau x_3 - x_3^\tau x_2 \in A(\tau)$.

Lemma 3.8. *With $A(\tau)$ as above, we have that $\tau \notin Aut(A(\tau))$, $\tau^2 \in Aut(A(\tau))$, Ω is normal and 1-regular in $A(\tau)$, and the Hilbert series of $A(\tau)^!$ is $H(x) = (1+x)^4$.*

Proof. Let

$$\omega' = -\eta_1^{\tau^{-2}}\eta_4^{\tau^{-1}} + \eta_2^{\tau^{-2}}\eta_3^{\tau^{-1}} + \frac{1}{2}\eta_1^{\tau^{-2}}\eta_1^{\tau^{-1}},$$

and observe that $(\omega')^\tau \notin k^\times\omega$. It may be verified by computation that $\omega^{\tau^2} \in k^\times\omega$, $\omega' \in k^\times\omega$ and ${\eta_i}^\theta\omega \in k^\times\omega'\eta_i$ for all i. It follows that $\tau \notin \mathrm{Aut}(A(\tau))$, $\tau^2 \in \mathrm{Aut}(A(\tau))$ and ω is normal in $S^!$. An argument analogous to the proof of [12, Lemma 4.9] proves that the Hilbert series of $A(\tau)^!$ is $H(x) = (1+x)^4$ and that ω is regular. By [10, Lemma 2.5], it follows that Ω is normal and 1-regular in $A(\tau)$. ■

Lemma 3.9. *For every $\tau \in Aut(Q)$ such that τ has a matrix representative of the form given in Lemma 3.7 (with respect to the η_i), the algebra $A(\tau)$ twists by a twisting system to $A(t)$ where t is given by the matrix*

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

with respect to the η_i.

Proof. We remark that $t^2 = 1$. The proof follows that of [12, Proposition 4.12].

We define a potential twisting system of $A(\tau)$ as follows. Let

$$t_n = \begin{cases} \tau^{-n} & \text{if } n \in 2\mathbb{Z} \\ \tau^{-n+1} \circ t_1 & \text{if } n \in 2\mathbb{Z}+1 \end{cases}$$

where $t_1 : A(\tau)_m \to A(\tau)_m$ is the linear map given by

$$t_1(a_1 \cdots a_m) = a_1^{t\tau^{-1}} a_2^{\tau(t\tau^{-1})\tau^{-1}} \cdots a_m^{\tau^{m-1}(t\tau^{-1})\tau^{-(m-1)}}$$

where $a_i \in A(\tau)_1$ for all i. By Lemma 3.8, we have $\tau^2 \in \mathrm{Aut}(A(\tau))$. By [12, Proposition 4.12], to prove that $\{t_n\}$ is a twisting system, it suffices to prove that t_1 is well defined on $A(\tau)$.

Viewing t_1 as a linear map on $T(V^*)$, we have that t_1 is bijective, and since $t^2 = 1$ we have

$$t_1((V^*)^i ab (V^*)^j) \subset \begin{cases} (V^*)^i(\tau^{-i}t_1\tau^i(ab))(V^*)^j & \text{if } i \in 2\mathbb{N}\cup\{0\} \quad (1) \\ (V^*)^i(\tau^{-i-1}t_1^{-1}\tau^{i-1}(ab))(V^*)^j & \text{if } i \in 2\mathbb{N}-1, \quad (2) \end{cases}$$

for all $a, b \in T(V^*)_1$, for all $j \in \mathbb{N}$. Thus it suffices to prove that the span I_2 of the defining relations of $A(\tau)$ is invariant under t_1, which is equivalent to proving that $(\tau^2 \circ t_1)(I_2) = I_2$. However, $(\tau^2 \circ t_1)(ab) = a^{t\tau}b^{\tau t}$, so $\tau^2 \circ t_1(x_i^\tau x_j - x_j^\tau x_i)$ vanishes on $\Gamma_\tau(Q)$ for all i, j. A computation verifies that the same holds for $\tau^2 \circ t_1(x_1^\tau x_1 - x_2^\tau x_3)$. It follows that $\tau^2 \circ t_1$ maps S_2 to S_2 and also $S_2^!$ to $S_2^!$ and is defined on $S_2^!$ by $(\tau^2 \circ t_1)(\eta\eta') = \eta^{\tau t}(\eta')^{t\tau}$ for all $\eta, \eta' \in S_1^!$. A computation shows that $(\tau^2 \circ t_1)(\omega) \in k^\times\omega'$, and $\omega' \in k^\times\omega$ by Lemma 3.8. It follows that $t_1(I_2) = I_2$, which completes the proof. ■

Theorem 3.10. *If R is a regular algebra whose point scheme P is the quadric Q, then R is isomorphic to an algebra $k[x_1, \dots, x_4]$ with defining relations*

$$(i)\quad \begin{array}{lll} x_1x_2 + cx_2x_1 = 0, & cx_2x_4 = x_4x_2, & c^{-1}x_3x_2 - cx_2x_3 = x_1x_4 + x_4x_1, \\ x_3x_1 + cx_1x_3 = 0, & x_3x_4 = cx_4x_3, & cx_2x_3 - c^{-1}x_3x_2 = 2(x_1^2 + cx_2x_3), \end{array}$$

or

$$(ii)\quad \begin{array}{lll} x_1x_2 = x_3x_1, & x_3x_4 = (x_1 + x_4)x_2, & x_1x_4 - x_4x_1 = x_1^2 - x_2^2 + x_3^2, \\ x_2x_1 = x_1x_3, & x_2x_4 = (x_1 + x_4)x_3, & x_1x_4 - x_4x_1 = -x_1^2 + 2x_3^2, \end{array}$$

where $c \in k^\times$. Moreover, for all $c \in k^\times$, the algebras given above are regular with Hilbert series $(1-x)^{-4}$ and have point scheme (Q, τ), where $Q = \mathcal{V}(x_1^2 - x_2x_3)$.

Proof. By the proof of [12, Theorem 4.13], it suffices to find one $A(\tau)$ in which Ω is central in order to prove that all $A(\tau)$ (where τ is given by Lemma 3.7) are regular. Let

$$T = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $i^2 = -1$. We have that $T \in \mathrm{Aut}(Q)$ and that T is a matrix of the type given in Lemma 3.7(i). The algebra $A(T)$ has defining relations

$$\begin{array}{lll} x_1x_2 = ix_2x_1, & x_3x_1 = ix_1x_3, & x_1x_4 + x_4x_1 = i(x_2x_3 + x_3x_2), \\ x_2x_4 = ix_4x_2, & x_4x_3 = ix_3x_4, & x_2x_3 + x_3x_2 = 2(ix_1^2 + x_2x_3), \end{array}$$

where $\Omega = -i(x_2x_3 + x_3x_2)$. A computation verifies that Ω is central in $A(T)$.

The defining relations of $A(\tau)$ are given in Lemma 3.5 where $\alpha = -2$ (by Lemma 3.6) and τ is given in Lemma 3.7. If τ is represented by a matrix in Lemma 3.7(i), then $A(\tau)$ has defining relations

$$\begin{array}{lll} x_1x_2 + cx_2x_1 = 0, & cx_2x_4 = (bx_1 + x_4)x_2, & c^{-1}x_3x_2 - cx_2x_3 = x_1x_4 + (bx_1 + x_4)x_1, \\ x_3x_1 + cx_1x_3 = 0, & x_3x_4 = c(bx_1 + x_4)x_3, & cx_2x_3 - c^{-1}x_3x_2 = 2(x_1^2 + cx_2x_3), \end{array}$$

where $c \in k^\times$ and $b \in k$. Mapping $x_i \mapsto x_i$ for $i \neq 4$ and $x_4 \mapsto x_4 - \frac{b}{2}x_1$ yields the defining relations given in (i), which give a one-parameter family of iterated Ore extensions. On the other hand, if τ is represented by a matrix in Lemma 3.7(ii), then $A(\tau)$ has defining relations

$$\begin{array}{lll} x_1x_2 = cx_3x_1, & cx_3x_4 = (bx_1 + x_4)x_2, & cx_3^2 - c^{-1}x_2^2 = x_1x_4 - (bx_1 + x_4)x_1, \\ x_2x_1 = cx_1x_3, & x_2x_4 = c(bx_1 + x_4)x_3, & cx_3^2 - c^{-1}x_2^2 = -2(x_1^2 - cx_3^2), \end{array}$$

for all $c \in k^\times$, $b \in k$. If $b = 0$, then τ is conjugate to a matrix given in Lemma 3.7(i) (with $b = 0$ and $c = 1$). So such an $A(\tau)$ is isomorphic to an algebra with defining relations given in (i) (where $c = 1$). If $b \neq 0$, then τ is conjugate to a matrix which is independent of b and c, so that all such $A(\tau)$ are isomorphic (since if $x, y \in A(\tau)_1$ and $\psi \in \mathrm{Aut}(\mathbb{P}^3)$, then $(x^\tau y)^\psi = (x^\psi)^{\psi^{-1}\tau\psi} y^\psi$). Hence we may take $b = c = 1$ in the matrix representing τ in Lemma 3.7(ii), which yields the defining relations given in (ii). ■

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