CORRIGENDUM TO "GENERALIZATIONS OF GRADED CLIFFORD ALGEBRAS AND OF COMPLETE INTERSECTIONS"

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Abstract. A correction is provided for Proposition 3.5 in the article "Generalizations of Graded Clifford Algebras and of Complete Intersections". The correction is: if S is a skew polynomial ring on finitely many generators of degree one that are normal elements in S, and if I is a homogeneous ideal of S that is generated by a normalizing sequence, then $\dim_{\Bbbk}(S/I)$ is finite if and only if S/I has no point modules and no fat point modules. A similar correction is provided for Corollary 3.6 of the same article.

The proof of Proposition 3.5 in [4] contains an error, so that [4, Proposition 3.5 and Corollary 3.6] need to be modified (see Proposition 10 and Corollary 11 below). The authors would like to thank J. T. Stafford for alerting them to this issue, which occurs in the paragraph in [4] immediately preceding Proposition 3.5. The main result of [4], namely Theorem 4.2, is correct as stated, provided that the definitions of *base point* and *base-point free* are changed from those given in [4, Definition 1.7] to those given in Definition 2 below. The examples and other results in the remaining sections of [4] are unchanged. Additionally, the reader should note that the results in [7] are unchanged.

We recall the notation of [4]: $\Bbbk$ denotes an algebraically closed field; $\Bbbk^\times = \Bbbk \setminus \{0\}$ and similarly for modules and other rings; $M_c(\Bbbk)$ is the ring of $c \times c$ matrices over $\Bbbk$; $\mu = (\mu_{ij}) \in M_n(\Bbbk^\times)$, where $\mu_{ij}\mu_{ji} = 1$ for all $i,\ j$; $S = \Bbbk\langle z_1, \ldots, z_n\rangle/\langle U\rangle$, where $U = \text{span}\{z_j z_i - \mu_{ij} z_i z_j\ :$

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$i,\ j = 1, \ldots, n,\ i \neq j\}$; $\mathcal{V}(U)$ denotes the zero locus of U in $\mathbb{P}(S_1^*) \times \mathbb{P}(S_1^*)$; a quadratic form q is any (nonzero) element of S_2 and its zero locus in $\mathcal{V}(U)$ is denoted $\mathcal{V}_{_U}(q)$; a quadric system is the span of a collection of quadratic forms; and $\Gamma = \{(p_1, p_2, \ldots) \in \mathbb{P}(S_1^*) \times \mathbb{P}(S_1^*) \times \cdots : (p_j, p_{j+1}) \in \mathcal{V}(U)$ for all $j \geq 1\}$.

Unless stated otherwise, we will work with right modules, and all the results will have left-sided analogues.

Definition 1. Let $B = \bigoplus_{i=0}^{\infty} B_i$ denote a connected, finitely generated, $\mathbb{N}$-graded $\Bbbk$-algebra that is generated by B_1.

(a) We define a *right base-point module* over B to be a graded right B-module M that is 1-critical with respect to GK-dimension (on nonzero graded submodules) such that $M = \bigoplus_{i=0}^{\infty} M_i = M_0 B$ and M has Hilbert series $H_M(t) = c/(1-t)$ for some $c \in \mathbb{N}$. (In fact, such a module will be 1-critical on all of its nonzero submodules.)

(b) [2, §3] In (a), if $c = 1$, then M is cyclic and M is called a (right) *point module.*

(c) [1, §3] In (a), if $c \geq 2$, then M is called a (right) *fat point module.*

By [2], if $\mathfrak{Q}$ is a normalizing quadric system, then the isomorphism classes of point modules over $S/\langle \mathfrak{Q} \rangle$ are parametrized by the scheme $\bigcap_{q \in \mathfrak{Q}} \mathcal{V}_{_U}(q)$.

Definition 2.

(a) (Replaces [4, Definition 1.7]) We define a *right base point* of a quadric system $\mathfrak{Q}$ to be any right base-point module over $S/\langle \mathfrak{Q} \rangle$, and we say a quadric system is *right base-point free* if it has no right base points. Similarly, for left base point, etc. We will see in Proposition 10 that a *normalizing* quadric system is right base-point free if and only if it is left base-point free.

(b) [4, Definition 3.3] If $I = \langle f_1, \ldots, f_m \rangle \subset S$, where $f_i \in S_{d_i}$, $d_i > 0$, for all $i = 1, \ldots, m$, then we say $p \in \mathcal{V}_{_U}(I)$ if $p = (p_1, p_2, \ldots) \in \Gamma$ and $f_i(p_1, \ldots, p_{d_i}) = 0$ for all i.

Remark 3. By [2], if a homogeneous ideal I of S is generated by a normalizing sequence, then the scheme $\mathcal{V}_{_U}(I)$ parametrizes the isomorphism classes of point modules over S/I. Otherwise, $\mathcal{V}_{_U}(I)$ parametrizes the isomorphism classes of point modules $M = \bigoplus_{i=0}^{\infty} M_i$ over S such that $M_0 f = 0$ for all $f \in I$.

If S is commutative, then S has no fat point modules (see Remark 8). Thus, by Remark 3, if S is commutative, then our definitions of “base point” and “base-point free” coincide with their commutative counterparts.

Notation 4. We write $\widehat{\mathcal{V}_{_U}(I)}$ to denote the set of isomorphism classes of right base-point modules over S/I.

Lemma 5. *Let B be as in Definition 1, and let $b \in B_d^{\times}$ denote a normal element of B, and suppose M is a base-point module over B. The element b acts faithfully on M if and only if $Mb \neq 0$.*

Proof. Suppose $Mb \neq 0$, and let $w \in M_i^\times$, for some i. Since b is normal, Mb is a submodule of M. However, M is a base-point module, so it follows that $(Mb)_j = M_j = (wB)_j$ for all $j \gg 0$. Hence, $M_{k+d} = (Mb)_{k+d} = M_k b = (wB)_k b$ for all $k \gg 0$. Thus, if $wb = 0$, then $M_{k+d} = \{0\}$ for all $k \gg 0$, since b is normal. This is a contradiction, so $wb \neq 0$ for all nonzero homogeneous elements $w \in M$. The result follows since M is graded and b is homogeneous. ■

Remark 6. Suppose $R = \bigoplus_{i=0}^{\infty} R_i$ is a noetherian, connected, $\mathbb{N}$-graded $\Bbbk$-algebra that is generated by R_1, and that z is a regular normal homogeneous element of R of positive degree. Let $R' = (R[z^{-1}])_0$, which is the subring of the localized ring $R[z^{-1}]$ that consists of the homogeneous elements of degree zero. If $M = \bigoplus_i M_i$ is a graded R-module on which z acts faithfully, then $(M[z^{-1}])_0$ is an R'-module. By [3, §7], this gives an equivalence between the category of graded R-modules $M = M_0R$ on which z acts faithfully such that $H_M(t) = c(1-t)^{-1}$ for some $c \in \mathbb{N}$ and the category of finite-dimensional R'-modules. In the reverse direction, one maps $N \mapsto \bigoplus_{i=0}^{\infty}(N \otimes_{R'} R[z^{-1}])_i$, where N is a finite-dimensional right R'-module. If $\deg(z) = 1$, then, as vector spaces, $(N \otimes_{R'} R[z^{-1}])_i \cong N \otimes_{\Bbbk} \Bbbk z^i$ for all $i \geq 0$.

The following lemma is a straightforward consequence of results in [3, §7], and is presumably well known; for lack of a suitable reference, we include a proof for completeness.

Lemma 7. *Let R, R' and z be as in Remark 6, and suppose* $\deg(z) = 1$.

(a) *If $M = \bigoplus_{i=0}^{\infty} M_i$ is a base-point module over R on which z acts faithfully, then M_i is a simple R'-module for all $i \geq 0$.*

(b) *If N is a finite-dimensional simple R'-module, then there exists a base-point module $\hat{N} = \bigoplus_{i=0}^{\infty} \hat{N}_i$ over R such that $\hat{N}_0 = N$ as R'-modules.*

Proof. (a) As i varies, $\dim_{\Bbbk}(M_i)$ is constant, so, for all i, the faithful action of z on M yields a bijective map $M_i \to M_{i+1}$, so M_i may be viewed as an R'-module. Suppose there exists a nonzero R'-module $N \subset M_i$ for some i. It follows that NR is an R-submodule of M and $NR = \sum_{k=0}^{\infty} Nz^k$. Since M is 1-critical, $(NR)_k = M_k$ for all $k \gg 0$. Hence, there exists $k \in \mathbb{N}$ such that $Nz^k = M_k$. As z acts faithfully on M, it follows that $\dim_{\Bbbk}(N) = \dim_{\Bbbk}(M_k) = \dim_{\Bbbk}(M_i)$, thereby giving $N = M_i$. Thus, M_i is a simple R'-module.

(b) By Remark 6, $\hat{N} = \bigoplus_{i=0}^{\infty}(N \otimes_{R'} R[z^{-1}])_i$ is a graded right R-module on which z acts faithfully, and $\hat{N} = \hat{N}_0 R$, $\hat{N}_0 = N$ and $H_{\hat{N}}(t) = c(1-t)^{-1}$, where $c = \dim_{\Bbbk}(N)$. By construction, $\hat{N}_i$ is an R'-module for all i. In fact, $\hat{N}_i$ is a simple R'-module for all i, since, if $e \in \hat{N}_i^\times$, then $e = v \otimes z^i$ for some $v \in N^\times$, and

$$eR' = (v \otimes z^i)R' = vR' \otimes \Bbbk z^i = N \otimes \Bbbk z^i = \hat{N}_i,$$

since N is a simple R'-module and z is normal and regular in R. It remains to prove that $\hat{N}$ is 1-critical. Let $M = \bigoplus_{i=d}^{\infty} M_i$ denote a nonzero graded R-submodule of $\hat{N}$, where $M_d \neq \{0\}$. Since z acts faithfully on $\hat{N}$, $M_k \neq \{0\}$ for all $k \geq d$ and $\{\dim_{\Bbbk}(M_k)\}_{k \geq d}$ is a nondecreasing sequence. However, $\dim_{\Bbbk}(\hat{N}_k) = c$ for all k, so there exists $j \in \mathbb{N}$ such that $\dim_{\Bbbk}(M_j) = \dim_{\Bbbk}(M_{j+1})$. Hence, as in the proof of (a), M_j is a nonzero R'-submodule of $\hat{N}_j$. It follows that $M_j = \hat{N}_j$, since $\hat{N}_j$ is a simple R'-module, and so $M_k = \hat{N}_k$ for all $k \gg 0$, whereby $\hat{N}$ is 1-critical. ■

Remark 8. By Lemma 5, if M is any base-point module over S, then z_i acts faithfully on M for some i. If, further, S is commutative, then invoking this discussion with Lemma 7(a) proves that the polynomial ring on finitely many generators does not have any fat point modules (by the nullstellensatz applied to $(S[z_i^{-1}])_0$ since $\Bbbk$ is algebraically closed).

Corollary 9. *Let R, R' and z be as in Lemma 7, and let $h_1, \ldots, h_\nu$ denote homogeneous elements of R of positive degree. Let $h_i' = h_i z^{-d_i} \in R'$, where $d_i = \deg(h_i)$ for all i. If*

$$0 < \dim_{\Bbbk}\left(\frac{R'}{\sum_{i=1}^{\nu} h_i' R'}\right) < \infty,$$

then there exists a base-point module M over R and a nonzero element $w \in M_0$ such that $wh_i = 0$ for all i.

Proof. The dimension hypothesis implies that the right R'-module $R'/(\sum_{i=1}^{\nu} h_i' R')$ maps onto a finite-dimensional simple R'-module N. Hence, there exists $w \in N^\times$ such that $wh_i' = 0$ for all i. By Lemma 7(b), there exists a base-point module M over R such that $M_0 = N$ as R'-modules. Hence, $wh_i = wh_i' z^{d_i} = 0$ for all i. ■

The next result replaces [4, Proposition 3.5]. We are grateful to J. T. Stafford for providing a counter-example to [4, Proposition 3.5] that motivated this corrigendum. That example has $n = 3$ and uses an ideal generated by a normalizing quadric system of dimension two; it can be obtained from Example 12 below by mapping $z_4 \mapsto 0$.

Proposition 10. (Replaces [4, Proposition 3.5]) *Let μ and S be as above, and let $I = \langle f_1, \ldots, f_\nu \rangle$, where $f_i \in S \setminus \Bbbk^\times$ is homogeneous for all i. If $\{f_1, \ldots, f_\nu\}$ is a normalizing sequence in S, then $\widehat{\mathcal{V}_U(I)}$ is empty if and only if $\dim_{\Bbbk}(S/I)$ is finite.*

Proof. Let $M = \bigoplus_{i=0}^{\infty} M_i \in \widehat{\mathcal{V}_U(I)}$. Since $M = M_0(S/I)$ and $\dim_{\Bbbk}(M_0) < \infty$, it follows that $\dim_{\Bbbk}(S/I) = \infty$ since $\dim_{\Bbbk}(M) = \infty$.

Conversely, suppose $\dim_{\Bbbk}(S/I) = \infty$. If $n = 1$, then S is commutative and $I = 0$, so the result holds. Henceforth, suppose inductively that $\widehat{\mathcal{V}_U(I)}$ is nonempty for all $n \le m - 1$. We will prove $\widehat{\mathcal{V}_U(I)}$ is nonempty for $n = m$.

If there exists k such that $\dim_{\Bbbk}\big(S/(I + \langle z_k \rangle)\big) = \infty$, then we may apply the induction hypothesis to $S/\langle z_k \rangle$ to prove there exists a base-point module M over $S/(I + \langle z_k \rangle)$. Since M is also a base-point module over S/I, it follows that $\widehat{\mathcal{V}_U(I)}$ is nonempty. Hence, we may assume $\dim_{\Bbbk}\big(S/(I + \langle z_k \rangle)\big) < \infty$ for all k and that $f_i \neq 0$ for all i.

It follows that there exists $t \in \mathbb{N}$ such that $z_i^t \in I + \langle z_k \rangle$ for all i, k. Since I is homogeneous and z_k is normal, there exist $g_{ik} \in S_{t-1}$ such that

$$z_i^t - g_{ik} z_k \in I, \tag{1}$$

for all i, k.

On the other hand, since $\dim_{\Bbbk}(S/I) = \infty$, $S_d \not\subset I$ for all d, so there exists $z \in \{z_1, \ldots, z_n\}$ such that $z^d \notin I$ for all $d \in \mathbb{N}$. By relabelling, we may assume $z = z_n$. Since z is normal and regular in S, we may form $S' = (S[z^{-1}])_0$. Owing to the relations of S, each generator of S', namely, $z_i' = z_i z^{-1}$ for all $i \neq n$, is normal in S'. In fact, S' has $\Bbbk$-basis

$$\{(z_1')^{i_1} \cdots (z_{n-1}')^{i_{n-1}} \;:\; i_k \in \mathbb{N} \cup \{0\} \text{ for all } k\}, \tag{2}$$

and we henceforth view $\deg(z_i') = 1$ for all i.

Let $f_i' = f_i z^{-d_i} \in S'$, where $d_i = \deg(f_i)$, for all i, and let $I' = \sum_{i=1}^{\nu} f_i' S'$. If $I' = S'$, then $1 \in I'$, so $z^t \in I$ for some $t \in \mathbb{N}$, which is false; thus $I' \neq S'$. Taking $z_k = z$ in (1) implies that there exists $g_i' \in S'$, of degree at most $t-1$, such that

$$(z_i')^t - g_i' \in I', \tag{3}$$

for all $i \neq n$. Since $\deg(g_i') \leq t-1$ for all $i \neq n$, and since z_i' is normal in S' for all i, (2) and (3) imply that S'/I' has a finite basis, whence $\dim_{\Bbbk}(S'/I') < \infty$.

By Corollary 9, there exists a base-point module M over S and $w \in M_0^{\times}$ such that $w f_i = 0$ for all i. By Lemma 5, $MI = 0$ since $\{f_1, \ldots, f_\nu\}$ is a normalizing sequence in S. Hence, $M \in \widehat{\mathcal{V}_U(I)}$, so $\widehat{\mathcal{V}_U(I)} \neq \emptyset$. ■

The next result is a revised version of [4, Corollary 3.6]; part (d) has been changed and the result has an extra sentence.

Corollary 11. (Replaces [4, Corollary 3.6]) *Let μ and S be as above. If $\{f_1, \ldots, f_n\}$ is a normalizing sequence in S of homogeneous elements of positive degree, then the following are equivalent:*

(a) *$\{f_1, \ldots, f_n\}$ is a regular sequence in S,*
(b) $\dim_{\Bbbk}(S/\langle f_1, \ldots, f_n\rangle) < \infty$,
(c) *for each $k = 1, \ldots, n$,* $\operatorname{GKdim}(S/\langle f_1, \ldots, f_k\rangle) = n - k$,
(d) *$\widehat{\mathcal{V}_U(I)}$ is empty, where $I = \langle f_1, \ldots, f_n\rangle$.*

Moreover, if $n \leq 3$ and $\deg(f_i) \leq 2$ for all i and $\dim_{\Bbbk}\left(\sum_{i=1}^n \Bbbk f_i\right) = n$, then $\widehat{\mathcal{V}_U(I)} = \mathcal{V}_U(I)$.

Proof. The equivalence of (a) through (d) follows from combining Proposition 10 with [4, Corollary 2.6]. For the second part, if $n \leq 2$, then S is a twist of the polynomial ring, and so S has no fat point modules, by Remark 8; the result follows in this case. Let $n = 3$. The result follows if $\deg(f_i) = 1$ for any i, so suppose that $\deg(f_i) = 2$ for all i and that $M = \bigoplus_{i=0}^{\infty} M_i \in \widehat{\mathcal{V}_U(I)}$. By Proposition 10, $\dim_{\Bbbk}(S/I) = \infty$, so we may assume $z^t \notin I$, for all $t \in \mathbb{N}$, where $z = z_3$. If z does not act faithfully on M, then Lemma 5 implies that M is a base-point module over $S/\langle z\rangle$, and so M is a point module.

Henceforth, we assume that z acts faithfully on M. Let $S' = (S[z^{-1}])_0$, and, for each i, we write $f_i' = f_i z^{-2}$. In particular, viewing the two generators of S' as having degree one, we have that $\deg(f_i') \leq 2$ for all i. We set $I' = \sum_{i=1}^{3} f_i' S'$, which is a two-sided ideal since $\{f_1', f_2', f_3'\}$ is normalizing in S'. As $z^t \notin I$, for all $t \in \mathbb{N}$, we have that $I' \neq S'$. Since $\dim_{\Bbbk}\left(\sum_{i=1}^{3} \Bbbk f_i\right) = 3$, we have that $\dim_{\Bbbk}\left(\sum_{i=1}^{3}\Bbbk f_i'\right) = 3$, and so $\dim_{\Bbbk}(S'/I') \leq 3$. On the other hand, since z acts faithfully on M, Lemma 7(a) implies that M_0 is a finite-dimensional simple S'/I'-module. In particular, $S'/\mathrm{Ann}(M_0) \hookrightarrow \mathrm{End}_{\Bbbk}(M_0)$, so $S'/\mathrm{Ann}(M_0)$ is a prime artinian ring, and so a simple ring. It follows (by the Artin-Wedderburn Theorem, since $\Bbbk$ is algebraically closed; c.f., [5]) that $S'/\mathrm{Ann}(M_0) \cong M_c(\Bbbk)$, where $c = \dim_{\Bbbk}(M_0)$. Hence, $3 \geq \dim_{\Bbbk}(S'/\mathrm{Ann}(M_0)) = c^2$, so $c = 1$, proving that M is a point module. ■

A consequence of Proposition 10 and Corollary 11 is that [4, Theorem 4.2] holds (using the new meaning of "base-point free" given in Definition 2 and replacing $\mathcal{V}_U(I)$ in the proof with $\widehat{\mathcal{V}_U(I)}$); moreover, if $n \leq 3$ in that theorem, then it holds with the original definition of base-point free given in [4].

The following example shows that if $n \geq 4$ in Corollary 11, then, in general, $\widehat{\mathcal{V}_U(I)} \neq \mathcal{V}_U(I)$, even if $\deg(f_i) = 2$ for all i and $\dim_{\Bbbk}\left(\sum_{i=1}^{n}\Bbbk f_i\right) = n$.

Example 12. Suppose $n = 4$ in Corollary 11. Let $\mu_{12} = -1 = \mu_{21}$ with all other $\mu_{ij} = 1$, and let $I = \langle \mathfrak{Q} \rangle$ where $\mathfrak{Q}$ is the normalizing quadric system given by $\{z_1^2 - z_2^2,\ z_1^2 - z_3^2,\ z_4^2,\ z_2 z_4\}$. Since $\dim_{\Bbbk}(\mathfrak{Q}) = n$, [4, Lemma 1.13] implies that this data corresponds to a graded skew Clifford algebra that is generated by four degree-one elements. Although $\mathcal{V}_U(I) = \emptyset$, the quadric system $\mathfrak{Q}$ is not base-point free since $\widehat{\mathcal{V}_U(I)}$ is nonempty. In particular, in the notation of the proof of Proposition 10, if $z = z_3$ (which is central) and $N = S'/(P + (z_2' - 1)S')$, where $P = \langle (z_1')^2 - 1, (z_2')^2 - 1, z_4' \rangle$, then the module $\bigoplus_{i=0}^{\infty}(N \otimes_{S'} S[z^{-1}])_i \in \widehat{\mathcal{V}_U(I)}$ and has Hilbert series $2(1-t)^{-1}$, since $\dim_{\Bbbk}(N) = 2$.

Fortunately, the examples of quadric systems in [4, §5], where $n = 4$, satisfy the definition of base-point free given in Definition 2, so the graded skew Clifford algebras given in those examples are regular, by [4, Theorem 4.2], as claimed. This may be verified directly for [4, §5.3], since the quadric system therein generates an ideal I of S that contains z_1^3, z_2^5, z_3^2 and z_4^5, so I has finite codimension in S. The verification for the examples in [4, §5.1 and §5.2] entails the observation that those quadric systems contain $z_1 z_2$ and $z_3 z_4$, so if the ideal I generated by the quadric system has infinite codimension in S, and if we invert z_4, then, using the notation from the proof of Proposition 10, the right ideal I' contains z_3' and $z_1' z_2'$, so I' is a two-sided ideal and S'/I' is commutative; similarly, if we instead invert z_j where $j \neq 4$. Thus, by Lemma 7(a) and the nullstellensatz applied to S'/I' (since $\Bbbk$ is algebraically closed), any base-point module over S/I is a point module.

Given the revision of [4, Corollary 3.6] in Corollary 11, [4, Definition 3.7] is modified accordingly as follows.

Definition 13. (Replaces [4, Definition 3.7]) Let μ be as above and let S be the $\Bbbk$-algebra on degree-1 generators $z_1, \ldots, z_n$ with defining relations $z_j z_i = \mu_{ij} z_i z_j$ for all i, j, where $i \neq j$. If $\{f_1, \ldots, f_n\}$ is a normalizing sequence in S of homogeneous elements of positive degree, then we call $S/\langle f_1, \ldots, f_n \rangle$ a *complete intersection* if any of Corollary 11(a)-(d) hold.

Hence, the analogous definition in the commutative setting is a special case of this last definition. Moreover, this definition is considered in [8] for algebras more general than S. The reader should perhaps also note that other notions of complete intersection exist in the literature, with most emphasizing a homological approach, such as the recent work in [6].

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