PRIMITIVE AND POISSON SPECTRA OF TWISTS OF POLYNOMIAL RINGS

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Abstract. A family of flat deformations of a commutative polynomial ring S on n generators is considered, where each deformation B is a twist of S by a semisimple, linear automorphism σ of $\mathbb{P}^{n-1}$, such that a Poisson bracket is induced on S. We show that if the symplectic leaves associated with this Poisson structure are algebraic, then they are the orbits of an algebraic group G determined by the Poisson bracket. In this case, we prove that if σ is "generic enough", then there is a natural one-to-one correspondence between the primitive ideals of B and the symplectic leaves if and only if σ has a representative in $\mathrm{GL}(\mathbb{C}^n)$ which belongs to G. As an example, the results are applied to the coordinate ring $\mathcal{O}_q(M_2)$ of quantum 2×2 matrices which is not a twist of a polynomial ring, although it is a flat deformation of one; if q is not a root of unity, then there is a bijection between the primitive ideals of $\mathcal{O}_q(M_2)$ and the symplectic leaves.

Introduction

Two non-commutative analogues of commutative algebras were introduced in [1, 2]: a twist of a commutative algebra by an automorphism, and a twisted homogeneous coordinate ring. In this paper, we consider the twist B of a polynomial algebra S on n generators over the complex field $\mathbb{C}$ by a semisimple automorphism σ of $\mathbb{P}^{n-1}$. In this case B may also be described as the twisted homogeneous coordinate ring $B(\mathbb{P}^{n-1}, \sigma, \mathcal{O}(1))$ where $\mathcal{O}(1)$ is the invertible sheaf of linear forms on $\mathbb{P}^{n-1}$. The algebra B is generally non-commutative; it is commutative if and only if σ is the identity map. However, due to the construction of B from $\sigma \in \mathrm{Aut}(\mathbb{P}^{n-1})$, there is a natural geometry associated to B; namely, that of $(\mathbb{P}^{n-1}, \sigma)$ or, equivalently, $(\mathbb{C}^n, \sigma)$.

In this paper we consider a family of twists $B(\mathfrak{m})$ of S, where the family is parametrized by the maximal ideals $\mathfrak{m}$ of a principal ideal domain R. The multiplication in the "generic" $B(\mathfrak{m})$ (which is determined by an automorphism $\sigma_{\mathfrak{m}}$ of $\mathbb{P}^{n-1}$) induces a Poisson bracket on S, and hence a Poisson structure on $\mathbb{C}^n$.

Our main objective is to analyse the interplay between the geometry of $(\mathbb{C}^n, \sigma_{\mathfrak{m}})$ and the Poisson geometry.

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The main results of the paper are Theorems 3.8 and 3.12 which prove that if the symplectic leaves of the Poisson structure are algebraic, then the primitive ideals of the generic $B(\mathfrak{m})$ are parametrized by the symplectic leaves if and only if $\sigma_{\mathfrak{m}}$ has a representative in $\mathrm{GL}(\mathbb{C}^n)$ which belongs to a certain algebraic group G whose orbits are the symplectic leaves. In particular, in this case, the symplectic leaves are $\sigma_{\mathfrak{m}}$-invariant.

This problem is motivated by the following well known classical and quantum examples. The primitive ideals of the universal enveloping algebra of an algebraic solvable Lie algebra $\mathfrak{g}$ are parametrized by the symplectic leaves in $\mathfrak{g}^*$, and the symplectic leaves are the orbits of an algebraic subgroup of $\mathrm{GL}(\mathfrak{g})$ (namely, the adjoint algebraic group of $\mathfrak{g}$) ([4]). In the early 1990's, Hodges and Levasseur proved in [7, 8] that the primitive ideals of the quantum group $\mathcal{O}_q(SL_n)$ are parametrized by the symplectic leaves in SL_n which are associated to the Poisson bracket induced on $\mathcal{O}(SL_n)$; this has since been extended by Joseph in [9, 10, 11] from SL_n to more arbitrary groups. Since some homomorphic images of $\mathcal{O}_q(SL_n)$ and other quantum groups are twists of commutative algebras or are twisted homogeneous coordinate rings, it seems reasonable to ask if similar results hold for such algebras.

In §1, we consider certain homogeneous Poisson structures on $\mathbb{C}^n$ induced from Poisson brackets on group rings, and define the algebraic group G (Definition 1.3) whose orbits are the minimal Poisson subvarieties. In Proposition 1.4, necessary and sufficient conditions are found under which the orbits of G are the symplectic leaves. The group G plays a role analagous to that of the adjoint algebraic group of an algebraic solvable Lie algebra – the analogy is made more precise in Remark 1.6.

The next section focuses on a family of twisted group rings $\mathbb{C}^{\mathfrak{m}}\Gamma$, where Γ is a free abelian group of rank n and $\mathfrak{m}$ is a maximal ideal of a principal ideal domain R. For each $\mathfrak{m}$, we define a group of automorphisms $\{\tau_{\mathfrak{m},\beta} : \mathbb{C}\Gamma \to \mathbb{C}\Gamma \mid \beta \in \Gamma\}$ which is determined by the multiplication of $\mathbb{C}^{\mathfrak{m}}\Gamma$. We prove in Proposition 2.7 that if $\tau_{\mathfrak{m},\beta} \in G$ for all $\beta \in \Gamma$, then the spectrum of the "generic" $\mathbb{C}^{\mathfrak{m}}\Gamma$ is homeomorphic to the Poisson spectrum of $\mathbb{C}\Gamma$.

The third section specialises §2 to a family of twists of a polynomial algebra S. We show that, in this case, if the Poisson bracket is nonzero, then it has rank two (Lemma 3.3), and conditions on the Poisson bracket are determined for the symplectic leaves to be algebraic (Corollary 3.4). The main results (stated above) appear in Theorems 3.8 and 3.12, after which we give some examples and apply the results to an algebra that is not a twist of a polynomial algebra; namely the coordinate ring $\mathcal{O}_q(M_2)$ of quantum 2×2 matrices, where $q \in \mathbb{C}^\times$ ([6]). The family $\mathcal{O}_q(M_2)$ induces a homogeneous Poisson bracket on the polynomial ring on four variables and hence a Poisson structure on $M_2 (\cong \mathbb{C}^4)$. Corollary 3.15 demonstrates that if q is not a root of unity, then the primitive ideals of $\mathcal{O}_q(M_2)$ are parametrized by the symplectic leaves in M_2.

We finish the paper by comparing the geometry of $(\mathbb{C}^n, \sigma_{\mathfrak{m}})$ with the Poisson geometry of $\mathbb{C}^n$, in the case of twists $B(\mathfrak{m})$ of S under the hypotheses of Theorem 3.12, where $\sigma_{\mathfrak{m}}$ has a representative in $\mathrm{GL}(\mathbb{C}^n)$ which belongs to G. In Remark 3.16 we observe that the varieties of $\mathbb{C}^n$ corresponding to the primitive ideals of $B(\mathfrak{m})$ are connected and $\sigma_{\mathfrak{m}}$-invariant, and that their irreducible components are cyclically permuted by $\sigma_{\mathfrak{m}}$.

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1. Poisson Geometry of $(\mathbb{C}^\times)^n$

Let $\mathbb{C}^\times$ denote $\mathbb{C} \setminus \{0\}$ and let S be the commutative free $\mathbb{C}$-algebra on n generators $x_1, \dots, x_n$. We define a grading on S by $\deg(x_i) = 1$ for all i; the vector space of homogeneous elements of degree m in S will be denoted S_m.

Fix a free Abelian group $(\Gamma, +)$ of rank n. The group ring $\mathbb{C}\Gamma = \mathbb{C}[x_\alpha : \alpha \in \Gamma]$ is isomorphic to the algebra $S[x_1^{-1}, \dots, x_n^{-1}]$ where $\alpha = (i_1, \dots, i_n) \leftrightarrow x_1^{i_1} \cdots x_n^{i_n} = x_\alpha$. Let $b : \Gamma \times \Gamma \to \mathbb{C}$ be a skew-symmetric bilinear form and define a Poisson bracket $\{\ ,\ \}_b$ on $\mathbb{C}\Gamma$ via

$$\{x_\alpha, x_\beta\}_b = b(\alpha, \beta) x_\alpha x_\beta$$

for all $\alpha, \beta \in \Gamma$. A Poisson ideal of $\mathbb{C}\Gamma$ is an ideal J of $\mathbb{C}\Gamma$ such that $\{J, \mathbb{C}\Gamma\} \subseteq J$.

Definition 1.1. Let $Z_b(\mathbb{C}\Gamma)$ denote the subalgebra $\{f \in \mathbb{C}\Gamma : \{f, g\}_b = 0 \text{ for all } g \in \mathbb{C}\Gamma\}$ of $\mathbb{C}\Gamma$, and let Γ_b denote the subgroup $\{\alpha \in \Gamma : b(\alpha, \beta) = 0 \text{ for all } \beta \in \Gamma\}$ of Γ.

Lemma 1.2.
(a) $Z_b(\mathbb{C}\Gamma) = \mathbb{C}\Gamma_b$
(b) If J is a Poisson ideal of $\mathbb{C}\Gamma$, then $J = \mathbb{C}\Gamma(J \cap Z_b(\mathbb{C}\Gamma))$.

Proof. Part (a) is an easy consequence of the definition. To prove (b) consider $\mathbb{C}\Gamma$ as a Γ-module via $\beta(x_\alpha) = b(\beta, \alpha) x_\alpha = \{x_\beta, x_\alpha\}_b x_{-\beta}$ for all $\alpha, \beta \in \Gamma$. As a Γ-module, $\mathbb{C}\Gamma = \bigoplus_{\alpha \in T} Z_b(\mathbb{C}\Gamma) x_\alpha$ where T is any transversal for Γ_b in Γ; each $Z_b(\mathbb{C}\Gamma) x_\alpha$, where $\alpha \in T$, is a weight space for this module action. If J is a Poisson ideal of $\mathbb{C}\Gamma$, then J is a Γ-submodule of $\mathbb{C}\Gamma$, and, as a sum of weight spaces,

$$J = \bigoplus_{\alpha \in T} J \cap (Z_b(\mathbb{C}\Gamma) x_\alpha) = \bigoplus_{\alpha \in T} (J \cap Z_b(\mathbb{C}\Gamma)) x_\alpha$$

since x_α is a unit in $\mathbb{C}\Gamma$. The result follows. ∎

Such a Poisson bracket on $\mathbb{C}\Gamma$ induces a Poisson structure on $(\mathbb{C}^\times)^n$; more precisely, the extension, $\bar{b}$, of b by scalars induces a symplectic form on $(\mathbb{C}^\times)^n$. In our situation, in which the Poisson bracket is homogeneous, the rank of the symplectic form at any point of $(\mathbb{C}^\times)^n$ is constant and is given by the rank of $\bar{b}$ at any point.

A symplectic leaf in $(\mathbb{C}^\times)^n$ is a maximal connected Poisson submanifold L of $(\mathbb{C}^\times)^n$ on which the symplectic form determined by b is nondegenerate; in particular, $\dim(L) = \mathrm{rank}(\bar{b})$, which is even since b is skew-symmetric. By standard theory (e.g., see [12]), $(\mathbb{C}^\times)^n$ is a disjoint union of its symplectic leaves. However, although the symplectic leaves need not be algebraic varieties, the maximal Poisson ideals of $\mathbb{C}\Gamma$ determine minimal Poisson subvarieties of $(\mathbb{C}^\times)^n$. Since each minimal Poisson subvariety is a union of symplectic leaves, any component of a minimal Poisson subvariety would be a Poisson subvariety, and so the minimal Poisson subvarieties of $(\mathbb{C}^\times)^n$ are connected.

A symplectic leaf is determined by a Poisson ideal in a suitable (possibly nonalgebraic) extension ring of $\mathbb{C}\Gamma$. We will call a symplectic leaf L in $(\mathbb{C}^\times)^n$ algebraic if $L = \mathcal{V}(J)$ where J is a Poisson ideal in $\mathbb{C}\Gamma$.

Definition 1.3. Let $G \subset (\mathbb{C}^\times)^n$ denote the zero locus in $(\mathbb{C}^\times)^n$ of the ideal of $\mathbb{C}\Gamma$ which is generated by all elements of the form $x_\alpha - 1$ where $\alpha \in \Gamma_b$.

The variety G is an algebraic group under coordinatewise multiplication. The following result shows G plays a role analogous to that of the adjoint algebraic group of an algebraic solvable Lie algebra.

Proposition 1.4. *The minimal Poisson subvarieties in $(\mathbb{C}^\times)^n$ are the orbits of G in $(\mathbb{C}^\times)^n$; they are the symplectic leaves if and only if Rad$(\bar{b}) = \mathbb{C}\otimes_{\mathbb{Z}}\Gamma_b$, where $\bar{b} : (\mathbb{C}\otimes_{\mathbb{Z}}\Gamma)\times(\mathbb{C}\otimes_{\mathbb{Z}}\Gamma) \to \mathbb{C}$ is extension of b by scalars.*

Proof. Since $\mathbb{C}$ is algebraically closed, the maximal ideals $\mathcal{M}$ of $Z_b(\mathbb{C}\Gamma)$ are generated by elements of the form $x_\alpha - \lambda_\alpha$ for all $\alpha \in \Gamma_b$ where $\lambda_\alpha \in \mathbb{C}^\times$ (by the Nullstellensatz). It follows from Lemma 1.2 that the maximal Poisson ideals of $\mathbb{C}\Gamma$ are generated by such $\mathcal{M}$. If $\rho \in G$ and $\mu \in \mathcal{V}(x_\alpha - \lambda_\alpha)$, then

$$(x_\alpha - \lambda_\alpha)(\rho \cdot \mu) = x_\alpha(\rho)x_\alpha(\mu) - \lambda_\alpha = 1\lambda_\alpha - \lambda_\alpha = 0,$$

which implies that the minimal Poisson subvarieties are invariant under G. Similarly, if $\mu, \nu \in \mathcal{V}(x_\alpha - \lambda_\alpha)$ where $\mu = (\mu_i)$, $\nu = (\nu_i)$, then the element $\rho = (\nu_i\mu_i^{-1}) \in \mathcal{V}(x_\alpha - 1)$ satisfies $\rho(\mu) = \nu$. Hence the orbits of G are the minimal Poisson subvarieties and they all have dimension $n{-}\mathrm{rank}(\Gamma_b)$.

The rank of the symplectic form at each point is constant, so in order to check its rank, it suffices to check the rank of $\bar{b}$ at one point $p \in (\mathbb{C}^\times)^n$. By identifying $\mathbb{C} \otimes_{\mathbb{Z}} \Gamma$ with the tangent space $T_p((\mathbb{C}^\times)^n)$ via $\alpha \mapsto (d/dx_\alpha)|_p$, the radical of the symplectic form on $T_p((\mathbb{C}^\times)^n)$ may be identified with the radical of $\bar{b}$; thus, $\mathbb{C} \otimes_{\mathbb{Z}} \Gamma_b \subseteq \mathrm{Rad}(\bar{b})$. Equality holds if and only if $\mathrm{rank}(\bar{b}) = n{-}\mathrm{rank}(\Gamma_b)$, which is the dimension of each minimal Poisson subvariety and so completes the proof. ■

If the symplectic leaves are algebraic, then they are the minimal Poisson subvarieties, in which case $\mathrm{Rad}(\bar{b}) = \mathbb{C} \otimes_{\mathbb{Z}} \Gamma_b$ and the dimension of each leaf is $n - \mathrm{rank}(\Gamma_b)$. In this situation, $\mathrm{rank}(\bar{b}) = n - \mathrm{rank}(\Gamma_b) = \dim(G)$.

If $b = 0$, then the symplectic leaves are given by the maximal ideals of $\mathbb{C}\Gamma$, and so are algebraic.

Corollary 1.5.

(a) *If $b = \mu b'$, where $\mu \in \mathbb{C}$ and b' takes values in $\mathbb{Q}$, then the symplectic leaves are algebraic.*

(b) *If rank$(\bar{b}) = 2$, then the symplectic leaves are algebraic if and only if $b = \mu b'$, where $\mu \in \mathbb{C}^\times$ and b' takes values in $\mathbb{Q}$.*

Proof. (a) Let $\bar{b}, \overline{b'} : (\mathbb{C} \otimes_{\mathbb{Z}} \Gamma) \times (\mathbb{C} \otimes_{\mathbb{Z}} \Gamma) \to \mathbb{C}$ denote extension of b and b' by scalars. Since b' takes values in $\mathbb{Q}$, there exists a basis for Γ with respect to which b' has matrix $M \in M_n(\mathbb{Q})$. However, $\mathrm{Rad}(\overline{b'})$ consists of all $v \in \mathbb{C} \otimes_{\mathbb{Z}} \Gamma$ such that $\overline{b'}(-, v) = 0$; that is, $Mv = 0$. To find all such $v \in \mathbb{C} \otimes_{\mathbb{Z}} \Gamma$, one may perform row operations on M, which yields that $\mathrm{Rad}(\overline{b'})$ is generated by all $v \in \mathbb{Q}^n$ such that $Mv = 0$ (since $M \in M_n(\mathbb{Q})$). By clearing denominators, we have that $\mathrm{Rad}(\overline{b'})$ is generated by all $v \in \mathbb{Z}^n$ such that $Mv = 0$; that is, $\mathrm{Rad}(\overline{b'}) = \mathbb{C} \otimes_{\mathbb{Z}} \Gamma_b$. The result follows by Proposition 1.4, since $\mathrm{Rad}(\bar{b}) = \mathrm{Rad}(\overline{b'})$ and $\Gamma_b = \Gamma_{b'}$ if $\mu \neq 0$.

(b) Suppose that the symplectic leaves are algebraic and that $\mathrm{rank}(\bar{b}) = 2$. Then, by Proposition 1.4, $\mathrm{rank}(\Gamma_b) = n - 2$. Since Γ/Γ_b is torsionfree, there exists a basis $\alpha_1, \ldots, \alpha_n$ for Γ such that $\alpha_3, \ldots, \alpha_n$ is a basis for Γ_b. It follows that $b(\alpha_1, \alpha_j) = 0$ for all $j \geq 3$, which forces $b(\alpha_1, \alpha_2) = \mu$ for some $\mu \in \mathbb{C}^\times$. Moreover, $b(\alpha_i, \alpha_j) = 0$ whenever $\{i, j\} \neq \{1, 2\}$, so that the matrix $(b(\alpha_i, \alpha_j))$ has only two nonzero entries, namely $\pm\mu$. Hence the form $b' = \mu^{-1} b$ takes values in $\mathbb{Z}$. The result follows by invoking part (a). ■

Remark 1.6. If $f \in \mathbb{C}\Gamma$, define $\mathrm{ad}\, f : \mathbb{C}\Gamma \to \mathbb{C}\Gamma$ by $(\mathrm{ad}\, f)(g) = \{f, g\}_b$ for all $g \in \mathbb{C}\Gamma$. Since $(\mathrm{ad}\, x_i)|_S \in S x_i$, it follows that $x_i^{-1}(\mathrm{ad}\, x_i)|_{S_1} \in \mathfrak{gl}(S_1)$. Let $\mathfrak{g}$ denote the subspace of $\mathfrak{gl}(S_1)$ spanned by $\{x_i^{-1}(\mathrm{ad}\, x_i)|_{S_1} : i = 1, \ldots, n\}$, which is an abelian Lie algebra under the usual Lie bracket of $\mathfrak{gl}(S_1)$. The smallest algebraic subgroup of $\mathrm{GL}(S_1)$ whose Lie algebra contains $\mathfrak{g}$ is isomorphic to the group G, since the former contains the group $\{\exp X : X \in \mathfrak{g}\}$ which is isomorphic to G (and G is algebraic by definition).

2. Families of Twisted Group Rings

Let R denote a commutative finitely generated $\mathbb{C}$-algebra which is a principal ideal domain but not a field, and let $U(R)$ denote the group of units of R. Let $\tau, \tau' : \Gamma \times \Gamma \to U(R)$ be bimultiplicative, skew-symmetric forms such that $\tau = (\tau')^2$, and consider the twisted group ring $R^\tau\Gamma = R[x_\alpha : \alpha \in \Gamma]$ which has defining relations

$$x_\alpha * x_\beta = \tau'(\alpha, \beta) x_{\alpha+\beta}$$

for all $\alpha, \beta \in \Gamma$. It follows that

$$x_\alpha * x_\beta = \tau(\alpha, \beta) x_\beta * x_\alpha,$$

for all $\alpha, \beta \in \Gamma$. We define a family of twisted group rings $\mathbb{C}^\mathfrak{m}\Gamma$ by

$$\mathbb{C}^\mathfrak{m}\Gamma = \frac{R^\tau\Gamma}{\langle \mathfrak{m} \rangle}$$

where $\mathfrak{m}$ is a maximal ideal of R. We assume that there exists a maximal ideal Rh of R such that $\tau(\alpha, \beta) \equiv 1$ modulo $\langle h \rangle$ for all $\alpha, \beta \in \Gamma$, so that $R^\tau\Gamma/\langle h \rangle \cong \mathbb{C}\Gamma$. Hence, each $\mathbb{C}^\mathfrak{m}\Gamma$ is a flat deformation of $\mathbb{C}\Gamma$.

The deformations $\mathbb{C}^\mathfrak{m}\Gamma$ induce a Poisson bracket on $\mathbb{C}\Gamma$ as follows. The hypotheses imply that if $f, g \in \mathbb{C}\Gamma$ and if $\tilde{f}, \tilde{g}$ are preimages of f, g in $R^\tau\Gamma$, then $\tilde{f} * \tilde{g} - \tilde{g} * \tilde{f} \in \langle h \rangle$, and for each $(\tilde{f}, \tilde{g})$ the multiple of h that arises in this way is well defined since $R^\tau\Gamma$ is a torsionfree R-module.

Definition 2.1. [5] If $f, g \in \mathbb{C}\Gamma$, then the Poisson bracket of f and g induced by the deformations $\mathbb{C}^\mathfrak{m}\Gamma$ is defined to be

$$\{f, g\} = \frac{\tilde{f} * \tilde{g} - \tilde{g} * \tilde{f}}{h} \quad \text{modulo } \langle h \rangle,$$

where $\tilde{f}, \tilde{g}$ denote preimages of f, g in $R^\tau\Gamma$.

This bracket is given by a skew-symmetric bilinear form as follows. Define $d\tau : \Gamma \times \Gamma \to \mathbb{C}$ by

$$d\tau(\alpha, \beta) = \frac{\tau(\alpha, \beta) - 1}{h} \quad \text{modulo } \langle h \rangle$$

for all $\alpha, \beta \in \Gamma$.

Lemma 2.2. *The form $d\tau$ is bilinear and skew-symmetric and determines the Poisson bracket on $\mathbb{C}\Gamma$ in Definition 2.1 via*

$$\{x_\alpha, x_\beta\} = d\tau(\alpha, \beta) x_\alpha x_\beta$$

for all $\alpha, \beta \in \Gamma$. ■

Let $\tau_\mathfrak{m} : \Gamma \times \Gamma \to R \to R/\mathfrak{m}$ denote the skew-symmetric bimultiplicative form corresponding to $\mathbb{C}^\mathfrak{m}\Gamma$.

Definition 2.3. Let Γ_τ (respectively, $\Gamma_{\mathfrak{m}}$) denote the subgroup of Γ consisting of those $\alpha \in \Gamma$ such that $\tau(\alpha, \beta) = 1$ (respectively, $\tau_{\mathfrak{m}}(\alpha, \beta) = 1$) for all $\beta \in \Gamma$.

Clearly, $\Gamma_\tau \subseteq \Gamma_{\mathfrak{m}}$ for all $\mathfrak{m}$.

Lemma 2.4.
(a) The centre of $\mathbb{C}^{\mathfrak{m}}\Gamma$ *is the group ring* $\mathbb{C}\Gamma_{\mathfrak{m}}$.
(b) If J *is an ideal of* $\mathbb{C}^{\mathfrak{m}}\Gamma$*, then* $J = \mathbb{C}^{\mathfrak{m}}\Gamma(J \cap (\mathbb{C}\Gamma_{\mathfrak{m}}))$.

Proof. Part (a) is an easy consequence of the definition. To prove (b), consider $\mathbb{C}^{\mathfrak{m}}\Gamma$ as a Γ-module via $\beta(x_\alpha) = \tau_{\mathfrak{m}}(\beta, \alpha)x_\alpha = x_\beta * x_\alpha * x_{-\beta}$ for all $\alpha, \beta \in \Gamma$. As a Γ-module, $\mathbb{C}^{\mathfrak{m}}\Gamma = \bigoplus_{\alpha \in T}(\mathbb{C}\Gamma_{\mathfrak{m}}) * x_\alpha$ where T is any transversal for $\Gamma_{\mathfrak{m}}$ in Γ; each $(\mathbb{C}\Gamma_{\mathfrak{m}}) * x_\alpha$, where $\alpha \in T$, is a weight space for this module action. If J is an ideal of $\mathbb{C}^{\mathfrak{m}}\Gamma$, then J is a Γ-submodule of $\mathbb{C}^{\mathfrak{m}}\Gamma$, and, as a sum of weight spaces,

$$J = \bigoplus_{\alpha \in T} J \cap ((\mathbb{C}\Gamma_{\mathfrak{m}}) * x_\alpha) = \bigoplus_{\alpha \in T} (J \cap \mathbb{C}\Gamma_{\mathfrak{m}}) * x_\alpha$$

since x_α is a unit in $\mathbb{C}^{\mathfrak{m}}\Gamma$. The result follows. ∎

Definition 2.5. We call $\mathfrak{m}$, or $\mathbb{C}^{\mathfrak{m}}\Gamma$, *generic* if $\Gamma_{\mathfrak{m}} = \Gamma_\tau$.

Since Γ has countable cardinality, there are at most countably many $\mathfrak{m}$ which are not generic. We aim to find conditions on $\tau_{\mathfrak{m}}$ which relate the Poisson ideals of $\mathbb{C}\Gamma$ and the prime ideals of $\mathbb{C}^{\mathfrak{m}}\Gamma$ for generic $\mathfrak{m}$.

For each $\beta \in \Gamma$ define the automorphism $\tau_{\mathfrak{m},\beta} \in \mathrm{Aut}(\mathbb{C}\Gamma)$ by

$$\tau_{\mathfrak{m},\beta}(x_\alpha) = \tau_{\mathfrak{m}}(\beta, \alpha)x_\alpha$$

for all $\alpha \in \Gamma$. We may view $\tau_{\mathfrak{m},\beta} \in (\mathbb{C}^\times)^n$; indeed, evaluating x_α at $\tau_{\mathfrak{m},\beta}$ yields $x_\alpha(\tau_{\mathfrak{m},\beta}) = \tau_{\mathfrak{m}}(\beta, \alpha)$ for all $\alpha \in \Gamma$.

Recall the algebraic group G from Definition 1.3 and that the bilinear form determining our Poisson bracket in this section is $d\tau$ (Lemma 2.2).

Lemma 2.6. *If* $\mathfrak{m}$ *is generic, then* $\Gamma_{\mathfrak{m}} = \Gamma_{d\tau}$ *if and only if* $\tau_{\mathfrak{m},\beta} \in G$ *for all* $\beta \in \Gamma$.

Proof. Since $\mathfrak{m}$ is generic, $\Gamma_{\mathfrak{m}} = \Gamma_\tau \subseteq \Gamma_{d\tau}$ by definition of $d\tau$. It follows that $\Gamma_{d\tau} = \Gamma_{\mathfrak{m}}$ if and only if $\tau_{\mathfrak{m}}(\beta, \alpha) = 1$ for all $\alpha \in \Gamma_{d\tau}$ for all $\beta \in \Gamma$. Hence $\Gamma_{d\tau} = \Gamma_{\mathfrak{m}}$ if and only if $x_\alpha(\tau_{\mathfrak{m},\beta}) = 1$ for all $\alpha \in \Gamma_{d\tau}$ for all $\beta \in \Gamma$, which completes the proof. ∎

The following result compares $\mathrm{Spec}(\mathbb{C}^{\mathfrak{m}}\Gamma)$ with the Poisson spectrum of $\mathbb{C}\Gamma$, where the latter is the set of Poisson prime ideals endowed with the topology of the Poisson ideals. (A Poisson prime ideal is a Poisson ideal P with the property that if I and J are Poisson ideals such that $IJ \subset P$, then $I \subset P$ or $J \subset P$; the closed sets of the Poisson spectrum are given by sets $\mathcal{V}(J) = \{\text{Poisson prime ideals } P : P \supset J\}$ where J is a Poisson ideal of $\mathbb{C}\Gamma$.)

Proposition 2.7. *If $\mathfrak{m}$ is generic and if $\tau_{\mathfrak{m},\beta} \in G$ for all $\beta \in \Gamma$, then Spec$(\mathbb{C}^{\mathfrak{m}}\Gamma)$ is homeomorphic to the Poisson spectrum of $\mathbb{C}\Gamma$ and the primitive ideals of $\mathbb{C}^{\mathfrak{m}}\Gamma$ are maximal; the homeomorphism is given by $I \mapsto (I \cap \mathbb{C}\Gamma_{\mathfrak{m}})\mathbb{C}\Gamma = (I \cap \mathbb{C}\Gamma_{d\tau})\mathbb{C}\Gamma$ where $I \subset \mathbb{C}^{\mathfrak{m}}\Gamma$ is primitive.*

Proof. By Lemma 2.6, we have $\tau_{\mathfrak{m},\beta} \in G$ for all $\beta \in \Gamma$ if and only if $\Gamma_{\mathfrak{m}} = \Gamma_{d\tau}$. Thus, the hypotheses, Lemma 1.2 and Lemma 2.4 imply that both the Poisson spectrum of $\mathbb{C}\Gamma$ and Spec$(\mathbb{C}^{\mathfrak{m}}\Gamma)$ are homeomorphic to the spectrum of the group ring $\mathbb{C}\Gamma_{\tau}$. The homeomorphism is given by $I \mapsto (I \cap \mathbb{C}\Gamma_{\mathfrak{m}})\mathbb{C}\Gamma = (I \cap \mathbb{C}\Gamma_{d\tau})\mathbb{C}\Gamma$ where $I \subset \mathbb{C}^{\mathfrak{m}}\Gamma$. Hence the maximal ideals of $\mathbb{C}^{\mathfrak{m}}\Gamma$ are homeomorphic to maximal Poisson ideals of $\mathbb{C}\Gamma$, and vice versa.

If P is a primitive ideal of $\mathbb{C}^{\mathfrak{m}}\Gamma$, then the centre of Fract$(\mathbb{C}^{\mathfrak{m}}\Gamma/P)$ is isomorphic to $\mathbb{C}$. It follows that $x_\alpha - \lambda_\alpha \in P$ for all $\alpha \in \Gamma_{\mathfrak{m}}$ for some $\lambda_\alpha \in \mathbb{C}$. Hence P is homeomorphic to a maximal Poisson ideal, so P is maximal. ■

The automorphisms $\tau_{\mathfrak{m},\beta}$ appear in [3, §4.5] and play a role in determining the cliques of prime ideals of $\mathbb{C}^{\mathfrak{m}}\Gamma$ providing the group generated by $\{\tau_{\mathfrak{m}}(\beta, \alpha) : \alpha, \beta \in \Gamma\}$ is torsionfree.

3. Families of Twists of Polynomial Rings

In this section, we specialise the theory of §2 to a certain subfamily of the algebras analysed in that section. The subfamily consists of $\mathbb{C}$-algebras such that each member is a "twist" (defined below) by a semisimple linear automorphism of the polynomial algebra S introduced in §1. In particular, each member is of the form $\mathbb{C}[x_1, \dots, x_n]$ with defining relations $x_i x_j = \alpha_{ij} x_j x_i$ for all i, j, where each $\alpha_{ij} \in \mathbb{C}$, $\alpha_{ii} = 1$ and $\alpha_{ij}\alpha_{j\ell}\alpha_{\ell i} = 1$ for all i, j, ℓ. Our main results are Theorems 3.8 and 3.12 which give necessary and sufficient conditions for the primitive ideals of the generic member of this subfamily to be in one-to-one correspondence with the symplectic leaves of the Poisson structure.

Formally, we define the subfamily as follows. Recall from §1 that S denotes the graded polynomial algebra over $\mathbb{C}$ on n variables and let R and $U(R)$ be defined as in §2. The natural degree function on S may be extended to the commutative R-algebra $R \otimes_{\mathbb{C}} S$, whereby we view $R \hookrightarrow R \otimes_{\mathbb{C}} S$ such that $\deg(R) = 0$. Let $\sigma \in \mathrm{Aut}(\mathbb{P}^{n-1}_R)$; that is, σ determines a graded, degree zero, R-algebra automorphism of $R \otimes_{\mathbb{C}} S$. We use x^σ to denote $\sigma(x)$ for all $x \in R \otimes_{\mathbb{C}} S$.

Definition 3.1. [1] The twist $(R \otimes_{\mathbb{C}} S)^\sigma$ of $R \otimes_{\mathbb{C}} S$ by the automorphism σ is defined to be the R-algebra which is determined by the two conditions:

(1) $(R \otimes_{\mathbb{C}} S)^\sigma = R \otimes_{\mathbb{C}} S$ as graded R-modules, and
(2) the multiplication $*$ in $(R \otimes_{\mathbb{C}} S)^\sigma$ is given by $x * y = x^{\sigma^j} y$ for all $x \in ((R \otimes_{\mathbb{C}} S)^\sigma)_i$, for all $y \in ((R \otimes_{\mathbb{C}} S)^\sigma)_j$, for all i, j (where the right hand side uses the multiplication of $R \otimes_{\mathbb{C}} S$).

We remark that if $\phi \in \operatorname{Aut}_R(R\otimes_{\mathbb{C}} S)$ such that $\phi|_{R\otimes_{\mathbb{C}} S_1} \in U(R)\sigma|_{R\otimes_{\mathbb{C}} S_1}$, then $(R\otimes_{\mathbb{C}} S)^\phi \cong (R\otimes_{\mathbb{C}} S)^\sigma$.

Henceforth, we additionally assume that σ has a representative which acts semisimply on $R\otimes_{\mathbb{C}} S_1$ in the sense that there exist generators $x_1,\dots,x_n \in S_1$ for $R\otimes_{\mathbb{C}} S$ which form a basis for $R\otimes_{\mathbb{C}} S_1$ as a free R-module such that $x_i^\sigma \in U(R)\otimes_{\mathbb{C}} x_i$ for all i. With these assumptions, the twist $(R\otimes_{\mathbb{C}} S)^\sigma$ has generators $x_1,\dots,x_n$ over R and defining relations:

$$x_i * x_j = r_{ij} x_j * x_i \quad \text{where} \quad r_{ij} \in U(R),\ r_{ii} = 1 \quad \text{and} \quad r_{ij} r_{j\ell} r_{\ell i} = 1 \tag{$*$}$$

for all i, j, ℓ. Moreover, it is straightforward to show that any R-algebra with generators $x_1,\dots,x_n$ over R and defining relations $(*)$ is a twist $(R\otimes_{\mathbb{C}} S)^\sigma$ for some $\sigma \in \operatorname{Aut}(R\otimes_{\mathbb{C}} S)$ which acts semisimply (in the above sense) on $R\otimes_{\mathbb{C}} S_1$. Clearly, $(R\otimes_{\mathbb{C}} S)^\sigma[x_1^{-1},\dots,x_n^{-1}]$ is a twisted group ring of a free abelian group, so the results of §2 may be applied.

For each maximal ideal $\mathfrak{m}$ of R, the $\mathbb{C}$-algebra

$$B(\mathfrak{m}) = \frac{(R\otimes_{\mathbb{C}} S)^\sigma}{\langle \mathfrak{m} \rangle},$$

is a twist of S by the induced map $\sigma_{\mathfrak{m}} \in \operatorname{Aut}(S)$. As in §2 we assume that there exists a maximal ideal Rh of R such that $\sigma \equiv$ identity modulo $\langle h \rangle$.

Remark 3.2. Since $\sigma \in \operatorname{Aut}(\mathbb{P}_R^{n-1})$, the algebra $B(\mathfrak{m})$, where $\mathfrak{m}$ is a maximal ideal of R, determines the automorphism $\sigma_{\mathfrak{m}}$ only up to a scalar multiple. In particular, if $f^{\sigma_{\mathfrak{m}}} \in \mathbb{C}^\times f$ for some $f \in \mathbb{C}\Gamma$, then the scalar multiple of f obtained in this way is only well defined if f is homogeneous of degree zero. For this reason, in the rest of the paper, we view $\sigma_{\mathfrak{m}} \in \operatorname{Aut}(\mathbb{P}^{n-1}_{R/\mathfrak{m}})$ where we identify $\mathbb{P}^{n-1}_{R/\mathfrak{m}}$ with $\mathbb{P}(S_1^*)$.

The family $\{B(\mathfrak{m})\}$ induces a Poisson bracket on S and on $\mathbb{C}\Gamma$, as defined in Definition 2.1 (and recall Lemma 2.2), which is determined by a nonzero skew-symmetric bilinear form $b : \Gamma\times\Gamma \to \mathbb{C}$. Since $(R\otimes_{\mathbb{C}} S)^\sigma[x_1^{-1},\dots,x_n^{-1}]$ is a twisted group ring $R^\tau\Gamma$, where $\tau : \Gamma\times\Gamma \to U(R)$ is the bimultiplicative skew-symmetric form determined by the multiplication in $(R\otimes_{\mathbb{C}} S)^\sigma[x_1^{-1},\dots,x_n^{-1}]$, we have that $b = d\tau$. Let $d\tau$ have matrix $(d\tau_{ij})$ with respect to the generators $x_1,\dots,x_n$; by $(*)$, we obtain $d\tau_{ii} = 0$ and $d\tau_{ij} + d\tau_{j\ell} + d\tau_{\ell i} = 0$ for all i, j, ℓ.

As mentioned in §1, the rank of the induced symplectic form at any point of $(\mathbb{C}^\times)^n$ is constant and is referred to as the rank of $\overline{d\tau}$, where $\overline{d\tau}$ is extension of $d\tau$ by scalars.

Lemma 3.3. *If $d\tau \neq 0$, then the rank of $\overline{d\tau}$ is two.*

Proof. With respect to the generators $x_1,\dots,x_n$, an arbitrary 3×3 submatrix of $(d\tau_{ij})$ has the form

$$\begin{bmatrix} d\tau_{sa} & d\tau_{sb} & d\tau_{sc} \\ d\tau_{ta} & d\tau_{tb} & d\tau_{tc} \\ d\tau_{ua} & d\tau_{ub} & d\tau_{uc} \end{bmatrix}, \tag{$\dagger$}$$

where $a, b, c, s, t, u \in \{1, \dots, n\}$. However, since $d\tau_{ij} + d\tau_{j\ell} + d\tau_{\ell i} = 0$ for all i, j, ℓ, subtracting the first column of (†) from each of the other two columns changes those columns into multiples of $(1,1,1)^T$. So the submatrix (†) has rank at most two. ■

Corollary 3.4. *The symplectic leaves in $(\mathbb{C}^\times)^n$ associated to the symplectic form $d\tau$ determined by $(R \otimes_{\mathbb{C}} S)^\sigma[x_1^{-1}, \dots, x_n^{-1}]$ are algebraic if and only if $d\tau = \mu\tilde{b}$ where $\tilde{b}$ takes values in $\mathbb{Q}$ and $\mu \in \mathbb{C}$.*

Proof. By Lemma 3.3 we have that $\operatorname{rank}(\overline{d\tau}) \leq 2$. The result follows from Corollary 1.5. ■

Recall from §2 that $\mathfrak{m}$ is said to be generic if $\Gamma_{\mathfrak{m}} = \Gamma_\tau$.

Proposition 3.5. *Suppose that $d\tau \neq 0$ and that the symplectic leaves in $(\mathbb{C}^\times)^n$ are algebraic.*

(a) *If $\alpha \in \Gamma_{d\tau}$, then $deg(x_\alpha) = 0$, and every Poisson ideal of $\mathbb{C}\Gamma$ is homogeneous.*
(b) *If $\mathfrak{m}$ is generic, then the ideals of $B(\mathfrak{m})[x_1^{-1}, \dots, x_n^{-1}]$ are homogeneous.*
(c) *If $deg(x_\alpha) = 0$ and $x_\alpha^{\sigma_\mathfrak{m}} = x_\alpha$, then $\alpha \in \Gamma_\mathfrak{m}$.*

Proof. (a) By Proposition 1.4 we have $\mathbb{C} \otimes_{\mathbb{Z}} \Gamma_{d\tau} = \operatorname{Rad}(\overline{d\tau})$, so, by Lemma 3.3, $\operatorname{rank}(\Gamma_{d\tau}) = \dim(\mathbb{C} \otimes_{\mathbb{Z}} \Gamma_{d\tau}) = n - 2$. By relabelling the x_i if necessary, we may assume that $d\tau_{12} \neq 0$.

With respect to the basis $\{x_i\}$, consider elements of the form $v = (v_i)$ where $v_i = d\tau_{2j}\delta_1^i + d\tau_{j1}\delta_2^i + d\tau_{12}\delta_j^i$ for all i, where $j \geq 3$ and δ denotes the Kronecker delta symbol. The dot product of row s of $(d\tau_{ij})$ with such an element v is

$$\begin{aligned} d\tau_{s1}d\tau_{2j} + d\tau_{s2}d\tau_{j1} + d\tau_{sj}d\tau_{12} &= d\tau_{s1}d\tau_{2j} + (d\tau_{s1} + d\tau_{12})(d\tau_{j2} + d\tau_{21}) + \\ &\qquad + (d\tau_{s1} + d\tau_{s2} + d\tau_{sj})d\tau_{12} \\ &= 0 \end{aligned}$$

for all s and for all $j \geq 3$, so $v \in \Gamma_{d\tau}$. Since such v are linearly independent and since $\operatorname{rank}(\Gamma_{d\tau}) = n - 2$, it follows that such v form a basis for $\Gamma_{d\tau}$. This proves the first part of (a), since the coordinates of v sum to zero.

The Poisson ideals of $\mathbb{C}\Gamma$ are generated by their intersection with the Poisson centre, $\mathbb{C}\Gamma_{d\tau}$, by Lemma 1.2, so (a) follows.

(b) Since $\Gamma_\tau \subseteq \Gamma_{d\tau}$, part (a) implies that $\deg(x_\alpha) = 0$ for all $\alpha \in \Gamma_\tau$. If $\mathfrak{m}$ is generic, then the same holds for all $\alpha \in \Gamma_\mathfrak{m}$, so the result follows by Lemma 2.4.

(c) Fix a representative of $\sigma_\mathfrak{m}$ in $\mathrm{GL}(\mathbb{C}^n)$ such that $x_i^{\sigma_\mathfrak{m}} = q_i x_i$ where $q_i \in \mathbb{C}^\times$ for all i. If $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n)$ where $\sum \alpha_i = 0$, then a computation shows that $\tau_\mathfrak{m}(\alpha, \beta) = (q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_n^{\alpha_n})^{\sum \beta_i}$. However, if $x_\alpha^{\sigma_\mathfrak{m}} = x_\alpha$, then $q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_n^{\alpha_n} = 1$, so $\tau_\mathfrak{m}(\alpha, \beta) = 1$. Since β was arbitrary, the proof is complete. ■

Recall the algebraic group G from Definition 1.3.

Lemma 3.6. *If $\Gamma_{d\tau} \subset \{\alpha \in \Gamma : deg(x_\alpha) = 0\}$, then $\sigma_\mathfrak{m}$ has a representative in $GL(\mathbb{C}^n)$ which belongs to G if and only if $\tau_{\mathfrak{m},\beta} \in G$ for all $\beta \in \Gamma$.*

Proof. Fix a representative of $\sigma_{\mathfrak{m}}$ in $\mathrm{GL}(\mathbb{C}^n)$ such that $x_i^{\sigma_{\mathfrak{m}}} = q_i x_i$ where $q_i \in \mathbb{C}^\times$ for all i. By viewing $\tau_{\mathfrak{m},\beta} \in (\mathbb{C}^\times)^n$ and $\sigma_{\mathfrak{m}} \in (\mathbb{C}^\times)^n$, a computation shows that

$$x_\alpha(\tau_{\mathfrak{m},\beta}) = (q_1^{\beta_1} \cdots q_n^{\beta_n})^{\sum \alpha_j} (x_\alpha(\sigma_{\mathfrak{m}}))^{-\sum \beta_i}$$

where $\alpha = (\alpha_j) \in \Gamma$ and $\beta = (\beta_i) \in \Gamma$. Hence, by our hypothesis, we have

$$x_\alpha(\tau_{\mathfrak{m},\beta}) = (x_\alpha(\sigma_{\mathfrak{m}}))^{-\sum \beta_i}$$

for all $\alpha \in \Gamma_{d\tau}$, for all $\beta \in \Gamma$. Thus, if $\sigma_{\mathfrak{m}} \in G$, then $\tau_{\mathfrak{m},\beta} \in G$ for all $\beta \in \Gamma$. Moreover, if $\tau_{\mathfrak{m},\beta} \in G$ for all $\beta \in \Gamma$, then, by choosing $\beta = (-1, 0, \ldots, 0) \in \Gamma$, we have that $x_\alpha(\sigma_{\mathfrak{m}}) = 1$ for all $\alpha \in \Gamma_{d\tau}$, which completes the proof. ■

By Proposition 2.7, it seems reasonable to compare the symplectic leaves in $(\mathbb{C}^\times)^n$ with the primitive ideals of the deformation $B(\mathfrak{m})[x_1^{-1}, \ldots, x_n^{-1}]$ of $\mathbb{C}\Gamma$. By Proposition 3.5 and Lemma 3.6, it follows that Proposition 2.7 shows that if the symplectic leaves are algebraic, then a sufficient condition to obtain a correspondence is to have a representative in $\mathrm{GL}(\mathbb{C}^n)$ of $\sigma_{\mathfrak{m}}$ which belongs to G; to have this condition also be necessary, we consider the homeomorphism in the proof of Proposition 2.7 from a different viewpoint, as outlined in the following remarks.

Remarks 3.7.

(1) Since $B(\mathfrak{m}) = S$ as graded vector spaces, an element $f \in B(\mathfrak{m})_i$ is an element of S_i and vice versa; for example, $B(\mathfrak{m})_2$ contains $x_i * x_j$ which is $x_i^\sigma x_j \in S_2$. Hence, a homogeneous left ideal I of $B(\mathfrak{m})$ is a homogeneous ideal of S, such that if $I = \sum_i B(\mathfrak{m}) * f_i$ where $f_i \in B(\mathfrak{m})_{j_i}$ $(= S_{j_i})$, then $I = \sum_i S f_i$.

(2) The preceding remark also applies to the twisted group rings $B(\mathfrak{m})[x_1^{-1}, \ldots, x_n^{-1}] = \mathbb{C}^{\mathfrak{m}}\Gamma$ and $S[x_1^{-1}, \ldots, x_n^{-1}] = \mathbb{C}\Gamma$. It follows that, for homogeneous ideals, the homeomorphism of Proposition 2.7 is the identification in the preceding remark between homogeneous ideals of $\mathbb{C}^{\mathfrak{m}}\Gamma$ and homogeneous ideals of $\mathbb{C}\Gamma$, where the elements of the ideals are the same.

(3) If I is an ideal of S (respectively, $S[x_1^{-1}, \ldots, x_n^{-1}]$), then we may associate to it a variety, denoted $\mathcal{V}(I)$, in $\mathbb{C}^n$ (respectively, $(\mathbb{C}^\times)^n$) which is the common locus of zeros of the elements of I.

Theorem 3.8. *Suppose that $\mathbb{C}^{\mathfrak{m}}\Gamma$ has the form $B(\mathfrak{m})[x_1^{-1}, \ldots, x_n^{-1}]$ and that $\mathfrak{m}$ is generic.*

(a) *If $\Gamma_{d\tau} \subseteq \{\alpha \in \Gamma : deg(x_\alpha) = 0\}$, then the primitive ideals of $\mathbb{C}^{\mathfrak{m}}\Gamma$ and the Poisson ideals of $\mathbb{C}\Gamma$ are homogeneous.*

(b) *If $\Gamma_{d\tau} \subseteq \{\alpha \in \Gamma : deg(x_\alpha) = 0\}$, then the correspondence in Remarks 3.7(1) defines a one-to-one correspondence between the primitive ideals of $\mathbb{C}^{\mathfrak{m}}\Gamma$ and the maximal Poisson ideals of $\mathbb{C}\Gamma$ if and only if $\sigma_{\mathfrak{m}}$ has a representative in $GL(\mathbb{C}^n)$ which belongs to G.*

(c) *If $\Gamma_{d\tau} \subseteq \{\alpha \in \Gamma : deg(x_\alpha) = 0\}$ or $d\tau = 0$, then the map $P \mapsto \mathcal{V}(P) \subset (\mathbb{C}^\times)^n$ defines a one-to-one correspondence between the primitive ideals of $\mathbb{C}^{\mathfrak{m}}\Gamma$ and the orbits of G in $(\mathbb{C}^\times)^n$ if and only if $\sigma_{\mathfrak{m}}$ has a representative in $GL(\mathbb{C}^n)$ which belongs to G.*

(d) *If the symplectic leaves in $(\mathbb{C}^\times)^n$ are algebraic, then the map $P \mapsto \mathcal{V}(P) \subset (\mathbb{C}^\times)^n$ defines a one-to-one correspondence between the primitive ideals of $\mathbb{C}^{\mathfrak{m}}\Gamma$ and the symplectic leaves in $(\mathbb{C}^\times)^n$ if and only if $\sigma_{\mathfrak{m}}$ has a representative in $GL(\mathbb{C}^n)$ which belongs to G.*

Proof. (a) Since $\Gamma_{\mathfrak{m}} = \Gamma_\tau \subseteq \Gamma_{d\tau} \subseteq \{\alpha \in \Gamma : \deg(x_\alpha) = 0\}$, the result follows from Lemma 2.4 and Lemma 1.2.

(b) By Proposition 2.7, Lemma 3.6, Remarks 3.7 and part (a), if $\sigma_{\mathfrak{m}} \in G$, then the correspondence holds. Conversely, if the correspondence holds, then we may view any primitive ideal of $\mathbb{C}^{\mathfrak{m}}\Gamma$ as containing the same elements as the corresponding maximal Poisson ideal of $\mathbb{C}\Gamma$. It follows that $\Gamma_{\mathfrak{m}} = \Gamma_{d\tau}$, by Lemma 2.4 and Lemma 1.2. Hence, by Lemma 2.6 and Lemma 3.6, we have $\sigma_{\mathfrak{m}} \in G$.

(c) By Proposition 1.4, if $d\tau \neq 0$, then this is part (b). If $d\tau = 0$, then $\Gamma_{d\tau} = \Gamma$, G is trivial and the orbits of G in $(\mathbb{C}^\times)^n$ are the points of $(\mathbb{C}^\times)^n$. Hence, in this case, the correspondence holds if and only if the primitive ideals correspond to the points of $(\mathbb{C}^\times)^n$. This happens if and only if $\mathbb{C}^{\mathfrak{m}}\Gamma$ is commutative, that is, if and only if $\sigma_{\mathfrak{m}}$ has a representative in $\mathrm{GL}(\mathbb{C}^n)$ which is the identity.

(d) If $d\tau = 0$, then this is (c). If $d\tau \neq 0$ and if the symplectic leaves are algebraic, then, by Proposition 3.5, $\Gamma_{d\tau} \subseteq \{\alpha \in \Gamma : \deg(x_\alpha) = 0\}$, and hence (d) follows from (c). ■

The Poisson bracket on S makes $\mathbb{C}^n$ a Poisson manifold, but the rank of the induced symplectic form on $\mathbb{C}^n$ is not constant; the rank is either two or zero at the points of $\mathbb{C}^n$. We call a symplectic leaf L in $\mathbb{C}^n$ algebraic if L, or its closure $\bar{L}$ (in the Zariski topology), is equal to $\mathcal{V}(J)$ where J is a Poisson ideal in S. A Poisson ideal of S which is the defining ideal of the closure of a symplectic leaf need not be a maximal Poisson ideal of S.

Since Poisson ideals of $\mathbb{C}\Gamma$ correspond to Poisson ideals of S which do not contain any monomial in the x_i and since maximal Poisson ideals of $\mathbb{C}\Gamma$ are prime, an algebraic symplectic leaf of $(\mathbb{C}^\times)^n$ is also an algebraic symplectic leaf of $\mathbb{C}^n$. If follows that if the symplectic leaves of $(\mathbb{C}^\times)^n$ are algebraic, then any symplectic leaf of $\mathbb{C}^n$ either lies on some $\mathcal{V}(x_i)$ or intersects none of the $\mathcal{V}(x_i)$.

The factor algebra $B(\mathfrak{m})/\langle x_{i_1}, \dots, x_{i_r}\rangle$, where $1 \leq r \leq n$, is the twist of the polynomial algebra on $n-r$ generators by the induced automorphism $\sigma'_{\mathfrak{m}} \in \mathrm{Aut}(S_1/(\mathbb{C}x_{i_1} + \cdots + \mathbb{C}x_{i_r}))$. It follows that

$$\frac{B(\mathfrak{m})}{\langle x_{i_1}, \dots, x_{i_r}\rangle}[x_j^{-1} : x_j \neq x_{i_\ell} \text{ for all } \ell \leq r]$$

is a twisted group ring $\mathbb{C}^{\mathfrak{m}}\Gamma'$ where Γ' is a subgroup of Γ of rank $n - r$.

Definition 3.9. Recall the notation from §2 concerning twisted group rings. For every subgroup $\Gamma' \subseteq \Gamma$, let τ' denote $\tau|_{\Gamma'\times\Gamma'}$, $(\tau')_{\mathfrak{m}}$ the restriction $\tau_{\mathfrak{m}}|_{\Gamma'\times\Gamma'}$ and $d\tau'$ the restriction $d\tau|_{\Gamma'\times\Gamma'}$. Define $(\Gamma')_{\tau'}$, $(\Gamma')_{\mathfrak{m}}$, $(\Gamma')_{d\tau'}$ and G' analogously to Γ_τ, $\Gamma_{\mathfrak{m}}$, $\Gamma_{d\tau}$ and G, and define

$$\Sigma = \{ \text{ maximal ideal } \mathfrak{m} \text{ of } R : (\Gamma')_{\tau'} = (\Gamma')_{\mathfrak{m}} \text{ for all subgroups } \Gamma' \text{ of } \Gamma\}.$$

Clearly, if $\mathfrak{m} \in \Sigma$, then $\mathfrak{m}$ is generic.

Lemma 3.10. *If the symplectic leaves of $(\mathbb{C}^\times)^n$ are algebraic, then so are the symplectic leaves of $\mathbb{C}^n$.*

Proof. By Corollary 3.4, if the symplectic leaves of $(\mathbb{C}^\times)^n$ are algebraic, then $d\tau = \mu\tilde{b}$ where $\mu \in \mathbb{C}$ and $\tilde{b}$ takes values in $\mathbb{Q}$. It follows that if Γ' is a subgroup of Γ, then $d\tau' = \mu\tilde{b}|_{\Gamma'\times\Gamma'}$. Hence, by Corollary 1.5, the symplectic leaves associated to $\mathbb{C}\Gamma'$ are algebraic for all $\Gamma' \subset \Gamma$. If Γ' is the subgroup generated by

$$\{(\alpha_j) \in \Gamma : \alpha_{i_\ell} = 0 \text{ for all } \ell \leq r\}$$

where $\operatorname{rank}(\Gamma') = n - r$, then such symplectic leaves correspond to Poisson ideals of S of the form $J + \langle x_{i_1}, \dots, x_{i_r}\rangle$ where J is an ideal of S such that $(J + \langle x_{i_1}, \dots, x_{i_r}\rangle)/\langle x_{i_1}, \dots, x_{i_r}\rangle$ does not contain any monomial. ■

Lemma 3.11. *Suppose that the symplectic leaves of $(\mathbb{C}^\times)^n$ are algebraic. If $\mathfrak{m} \in \Sigma$, then $\sigma_{\mathfrak{m}}$ has a representative in $GL(\mathbb{C}^n)$ which belongs to G if and only if $\sigma'_{\mathfrak{m}}$ has a representative which belongs to G' for every subgroup $\Gamma' \subseteq \Gamma$.*

Proof. It suffices to prove that if $\sigma_{\mathfrak{m}}$ has a representative in $\mathrm{GL}(\mathbb{C}^n)$ which belongs to G, then $\sigma'_{\mathfrak{m}}$ has a representative which belongs to G' for every subgroup Γ' of Γ. Suppose that $\mathfrak{m} \in \Sigma$ and that $\sigma_{\mathfrak{m}}$ has a representative in $\mathrm{GL}(\mathbb{C}^n)$ which belongs to G.

If $d\tau = 0$, then G is trivial, so $\sigma_{\mathfrak{m}}$ is the identity on $\mathbb{P}^{n-1}_{R/\mathfrak{m}}$ and hence $\sigma'_{\mathfrak{m}}$ is the identity on $\mathbb{P}^{n-r-1}_{R/\mathfrak{m}}$ for all subgroups Γ' of rank $n-r$ where $1 \leq r \leq n-1$ (recall Remark 3.2); thus, in this case, $\sigma'_{\mathfrak{m}}$ has a representative which belongs to G'.

Suppose that $d\tau \neq 0$ and that $\operatorname{rank}(\Gamma') = n-1$. In particular, by Lemma 2.6, Proposition 3.5 and Lemma 3.6, we have $\Gamma_{\mathfrak{m}} = \Gamma_{d\tau}$ since $\sigma_{\mathfrak{m}}$ has a representative in $\mathrm{GL}(\mathbb{C}^n)$ which belongs to G. Let $W = (\mathbb{C} \otimes_{\mathbb{Z}} \Gamma_{d\tau}) \cap (\mathbb{C} \otimes_{\mathbb{Z}} \Gamma')$, which is a subspace of $\mathbb{C} \otimes_{\mathbb{Z}} (\Gamma')_{d\tau'}$ and has dimension $n-3$ or $n-2$. If $d\tau' \neq 0$, then, by Proposition 1.4, Lemma 3.3 and Lemma 3.10, we have $\operatorname{rank}((\Gamma')_{d\tau'}) = n-3$, so $\dim(W) = n-3$. It follows that $W = \mathbb{C} \otimes_{\mathbb{Z}} (\Gamma')_{d\tau'}$, so $\mathbb{C} \otimes_{\mathbb{Z}} (\Gamma')_{d\tau'} \subseteq \mathbb{C} \otimes_{\mathbb{Z}} \Gamma_{d\tau} = \operatorname{rad}(\overline{d\tau})$. Thus, $(\Gamma')_{d\tau'} \subseteq \operatorname{rad}(\overline{d\tau}) \cap \Gamma$, so $(\Gamma')_{d\tau'} \subseteq \Gamma_{d\tau}$. Hence $(\Gamma')_{d\tau'} \subseteq \Gamma_{d\tau} \cap \Gamma' = \Gamma_{\mathfrak{m}} \cap \Gamma' \subseteq (\Gamma')_{\mathfrak{m}}$. However, since $\mathfrak{m} \in \Sigma$, we have $(\Gamma')_{\mathfrak{m}} = (\Gamma')_{\tau'} \subseteq (\Gamma')_{d\tau'}$, so equality holds in this case. It follows that $\sigma'_{\mathfrak{m}}$ has a representative which belongs to G' by Lemma 2.6, Proposition 3.5 and Lemma 3.6. On the other hand, if $d\tau' = 0$, then $\Gamma' = (\Gamma')_{d\tau'}$

and $\dim(W) = n-2$ (since $d\tau \neq 0$). So $\mathbb{C} \otimes_{\mathbb{Z}} \Gamma_{d\tau} \subseteq \mathbb{C} \otimes_{\mathbb{Z}} \Gamma'$, so that

$$\mathbb{C} \otimes_{\mathbb{Z}} \Gamma' = \bigoplus_{i=1}^{n-2} \mathbb{C}\alpha_i \oplus \mathbb{C}\alpha$$

where the $\alpha_i \in \Gamma_{d\tau}$ and $\alpha \in (\mathbb{C} \otimes_{\mathbb{Z}} \Gamma') \setminus (\mathbb{C} \otimes_{\mathbb{Z}} \Gamma_{d\tau})$. However, $\Gamma_{d\tau} = \Gamma_{\mathfrak{m}}$ and $\tau(\alpha, \alpha) = 1$, so by extending τ' to $(\mathbb{C} \otimes_{\mathbb{Z}} \Gamma') \times (\mathbb{C} \otimes_{\mathbb{Z}} \Gamma')$ via $\tau'(m_1 a, m_2 b) = \tau'(a, b)^{m_1 m_2}$, where $m_1, m_2 \in \mathbb{C}$, $a, b \in \Gamma'$, we find that $\tau'(\Gamma' \times \Gamma') = 1$, so $(\Gamma')_{\mathfrak{m}} = \Gamma'$. It follows that $\sigma'_{\mathfrak{m}}$ is the identity on $\mathbb{P}^{n-2}_{R/\mathfrak{m}}$, and hence has a representative which belongs to G'. The result now follows from reverse induction on the value of n. ■

Theorem 3.12. *Suppose that the symplectic leaves in $(\mathbb{C}^\times)^n$ are algebraic and that $\mathfrak{m} \in \Sigma$.*

(a) *The nonmaximal primitive ideals of $B(\mathfrak{m})$ are homogeneous.*

(b) *The map $P \mapsto \mathcal{V}(P) \subset \mathbb{C}^n$ defines a one-to-one correspondence between the primitive ideals of $B(\mathfrak{m})$ and the symplectic leaves in $\mathbb{C}^n$ if and only if $\sigma_{\mathfrak{m}}$ has a representative in $GL(\mathbb{C}^n)$ which belongs to G.*

Proof. By Theorem 3.8, the primitive ideals of $B(\mathfrak{m})$ which do not contain any x_i correspond to the symplectic leaves of $(\mathbb{C}^\times)^n$ if and only if $\sigma_{\mathfrak{m}}$ has a representative in $\mathrm{GL}(\mathbb{C}^n)$ which belongs to G. Since each $\langle x_i \rangle$ is a Poisson ideal, we may apply Theorem 3.8 (together with Lemma 3.10 and Lemma 3.11) inductively to $B(\mathfrak{m})/\langle x_{i_1}, \ldots, x_{i_r} \rangle$ for all $r \leq n$ to yield the result. ■

Example 3.13. Suppose that $n = \mathrm{rank}(\Gamma) = 4$ and that $R = \mathbb{C}[h][(1-h)^{-1}]$, where h is an indeterminate. Define $\sigma \in \mathrm{Aut}(R \otimes_{\mathbb{C}} S)$ by $x_1^\sigma = x_1$, $x_2^\sigma = (1-h)^{-1} x_2$, $x_3^\sigma = (1-h)^{-1} x_3$ and $x_4^\sigma = (1-h)^{-2} x_4$, so that $(R \otimes_{\mathbb{C}} S)^\sigma = R[x_1, \ldots, x_4]$ with six defining relations

$$\begin{array}{lll} x_1 * x_2 = (1-h) x_2 * x_1, & x_2 * x_4 = (1-h) x_4 * x_2, & x_2 * x_3 = x_3 * x_2, \\ x_1 * x_3 = (1-h) x_3 * x_1, & x_3 * x_4 = (1-h) x_4 * x_3, & x_1 * x_4 = (1-h)^2 x_4 * x_1. \end{array}$$

The Poisson bracket induced on S is determined by

$$\begin{array}{lll} \{x_1, x_2\} = -x_1 x_2, & \{x_2, x_4\} = -x_2 x_4, & \{x_2, x_3\} = 0, \\ \{x_1, x_3\} = -x_1 x_3, & \{x_3, x_4\} = -x_3 x_4, & \{x_1, x_4\} = -2 x_1 x_4, \end{array}$$

so the symplectic leaves are algebraic by Corollary 3.4. It therefore follows from Proposition 3.5 that $\deg(x_\alpha) = 0$ for all $\alpha \in \Gamma_{d\tau}$, so that $G = \mathcal{V}(x_2 - x_3, x_2^2 - x_1 x_4) \cap (\mathbb{C}^\times)^4$. The maximal ideals of R have the form $\langle h - q \rangle$ where $q \in \mathbb{C}$, $q \neq 1$, so $\sigma_{\mathfrak{m}}$ has a representative in $\mathrm{GL}(\mathbb{C}^n)$ which belongs to G for all $\mathfrak{m}$, and $\mathfrak{m} \in \Sigma$ if and only if $q - 1$ is not a root of unity, which holds if and only if $\mathfrak{m}$ is generic. Hence by Theorem 3.12 there is a one-to-one correspondence between the primitive ideals of the generic deformation $B(\mathfrak{m})$ and the symplectic leaves in $\mathbb{C}^4$.

In this example, the Poisson ideal $\langle x_1 x_4 - x_2 x_3 \rangle$ of S is $\sigma_{\mathfrak{m}}$-invariant. The corresponding factor algebra of $B(\mathfrak{m})$, where $\mathfrak{m} = \langle h - 1 + q \rangle$, is $A(\mathfrak{m}) = B(\mathfrak{m})/\langle x_1 * x_4 - q x_2 * x_3 \rangle$, which is isomorphic to a factor algebra of the coordinate ring of quantum 2×2 matrices (see Corollary

3.15) and also to a twist by an automorphism of a factor algebra of the coordinate ring of quantum symplectic 4-dimensional space.

Example 3.14. Let $R = \mathbb{C}[h][(1+a_i h)^{-1} : i = 1, \dots, n]$ where the $a_i \in \mathbb{C}$ are distinct for all i. Define $\sigma \in \mathrm{Aut}(R \otimes_{\mathbb{C}} S)$ by $x_i^\sigma = (1 + a_i h)x_i$ for all i. Then the twist $(R \otimes_{\mathbb{C}} S)^\sigma$ is the R-algebra generated by $x_1, \dots, x_n$ with defining relations $x_i * x_j = r_{ij} x_j * x_i$ for all i, j, where the $r_{ij} = (1 + a_i h)(1 + a_j h)^{-1}$. In $(R \otimes_{\mathbb{C}} S)^\sigma$ we have

$$x_i * x_j - x_j * x_i = h(a_i - a_j)(1 + a_i h)^{-1} x_i * x_j$$

for all i, j. Hence the induced Poisson bracket on S is determined by

$$\{x_i, x_j\} = (a_i - a_j)x_i x_j$$

for all i, j.

By Corollary 3.4, the symplectic leaves are algebraic if and only if there exists $\mu \in \mathbb{C}$ such that $a_i - a_j \in \mathbb{Q}\mu$ for all i, j. Moreover, if $\mathfrak{m} = \langle h - q \rangle$, then a calculation shows that $\mathfrak{m} \in \Sigma$ if and only if the subgroup of $\mathbb{C}^\times$ generated by $\{1 + a_i q : i = 1, \dots, n\}$ has torsionfree rank $\geq n - 1$ since the a_i are distinct. If the symplectic leaves are algebraic, then $\deg(x_\alpha) = 0$ for all $\alpha \in \Gamma_{d\tau}$ (by Proposition 3.5), so $(x_\alpha)^{\sigma_{\mathfrak{m}}} x_{-\alpha} \in \mathbb{C}$ is well defined for all $\alpha \in \Gamma_{d\tau}$, no matter which representative of $\sigma_{\mathfrak{m}} \in \mathrm{Aut}(\mathbb{P}^{n-1}_{R/\mathfrak{m}})$ is used (see Remark 3.2). In this case, a calculation shows that $\sigma_{\mathfrak{m}}$ has a representative in $\mathrm{GL}(\mathbb{C}^n)$ which belongs to G if and only if $x_\alpha^{\sigma_{\mathfrak{m}}} x_{-\alpha} = 1$ for all $\alpha \in \Gamma_{d\tau}$, which is true if and only if $\prod_{i=1}^{n} (1 + a_i q)^{\alpha_i} = 1$ for all $\alpha = (\alpha_1, \dots, \alpha_n) \in \Gamma_{d\tau}$.

If we take $n = 3$ and $a_i = i - 1$ for $i = 1, 2, 3$, then the symplectic leaves are algebraic, and $\mathfrak{m} = \langle h - q \rangle \in \Sigma$ if and only if the subgroup of $\mathbb{C}^\times$ generated by $\{1+q, 1+2q\}$ has torsionfree rank equal to two. However, $\Gamma_{d\tau} = \mathbb{Z}(-1, 2, -1)$, so $G = \mathcal{V}(x_2^2 - x_1 x_3) \cap (\mathbb{C}^\times)^3$, which yields that $\sigma_{\mathfrak{m}}$ has a representative in $\mathrm{GL}(\mathbb{C}^n)$ which belongs to G if and only if $\mathfrak{m} = \langle h \rangle$, which is not in Σ. Hence the desired correspondence fails. In fact, the maximal Poisson ideals are $\langle a x_1 x_3 + b x_2^2 \rangle$ for all $a, b \in \mathbb{C}^\times$ and

$$\langle x_i \rangle, \qquad \langle x_{i_1}, x_{i_2}, x_{i_3} - c \rangle \tag{‡}$$

for all i, for all $c \in \mathbb{C}$ and for all distinct i_1, i_2, i_3. If $\mathfrak{m} \in \Sigma$, then the primitive ideals of $B(\mathfrak{m})$ are those in (‡) and 0. So, in this example, not only does the correspondence fail, but, in addition, neither list of ideals is contained in the other.

As a corollary to Example 3.13, we may use Theorem 3.12 to prove that there is a one-to-one correspondence between the primitive ideals of the generic coordinate ring $\mathcal{O}_q(M_2)$ of quantum 2×2 matrices and the symplectic leaves in M_2. In [6], $\mathcal{O}_q(M_2)$ is defined to be a $\mathbb{C}$-algebra on four generators a, b, c, d with six defining relations

$$\begin{aligned} &ab = qba, \qquad bd = qdb, \qquad bc = cb, \\ &ac = qca, \qquad cd = qdc, \qquad ad - da = (q - q^{-1})bc, \end{aligned}$$

where $q \in \mathbb{C}^\times$. If $q = 1$, then $\mathcal{O}_1(M_2) \cong \mathcal{O}(M_2)$, a polynomial ring. It is well known that $\mathcal{O}_q(M_2)$ is a flat deformation of $\mathcal{O}(M_2)$ such that $M_2\,(\cong \mathbb{C}^4)$ carries the structure of a Poisson manifold. By [13, 14], $\mathcal{O}_q(M_2)$ is neither a twist of a polynomial ring nor a twisted homogeneous coordinate ring.

Corollary 3.15. *If q is not a root of unity, then the primitive ideals of $\mathcal{O}_q(M_2)$ are in one-to-one correspondence with the symplectic leaves in M_2.*

Proof. If q is not a root of unity, then the centre of $\mathcal{O}_q(M_2)$ is $\mathbb{C}[\Omega_q]$ where $\Omega_q = ad - qbc$. Thus (by standard theory) a primitive ideal of $\mathcal{O}_q(M_2)$ contains $\Omega_q - \lambda$ for some $\lambda \in \mathbb{C}$. The primitive ideals which contain Ω_q correspond to primitive ideals of the algebra $A(\mathfrak{m})$ from Example 3.13. It follows that if q is not a root of unity, then the primitive ideals which contain Ω_q are in one-to-one correspondence with the symplectic leaves in $\mathcal{V}(\Omega_1) \subset M_2$.

On the other hand, if $\lambda \neq 0$, then the automorphism $\phi : \mathcal{O}_q(M_2) \to \mathcal{O}_q(M_2)$ defined by $\phi(a) = \lambda a$, $\phi(b) = \lambda b$, $\phi(c) = c$, $\phi(d) = d$ takes $\Omega_q - \lambda$ to $\lambda(\Omega_q - 1)$. The primitive ideals which contain $\Omega_q - 1$ correspond to primitive ideals of $\mathcal{O}_q(SL_2)$ and, in [7], it is shown that they are in one-to-one correspondence with the symplectic leaves in $SL_2 = \mathcal{V}(\Omega_1 - 1) \subset M_2$ if q is not a root of unity. The result follows. ■

Remark 3.16. Suppose that the hypotheses of Theorem 3.12 hold and that $\sigma_{\mathfrak{m}}$ has a representative in $\mathrm{GL}(\mathbb{C}^n)$ which belongs to G. Then every nonmaximal primitive ideal P of $B(\mathfrak{m})$ is homogeneous, and $\mathcal{V}(P)$, the closure of the corresponding symplectic leaf in $\mathbb{C}^n$, is connected (but not necessarily irreducible). Since the symplectic leaves in $(\mathbb{C}^\times)^n$ are the orbits of G, they are $\sigma_{\mathfrak{m}}$-invariant. It follows from Lemma 3.10 and Lemma 3.11 that the closures of the symplectic leaves in $\mathbb{C}^n$ are $\sigma_{\mathfrak{m}}$-invariant. In fact, we claim that $\sigma_{\mathfrak{m}}$ cyclically permutes the irreducible components of $\mathcal{V}(P)$ where P is a primitive ideal of $B(\mathfrak{m})$. This can be seen as follows.

If P is maximal, then, after factoring out suitable x_i, P corresponds to a maximal Poisson ideal P^o in a suitable polynomial ring such that $\mathcal{V}(P) \cong \mathcal{V}(P^o)$. Since P^o is prime, $\mathcal{V}(P)$ is irreducible, so has no proper irreducible components. On the other hand, if P is a nonmaximal primitive ideal of $B(\mathfrak{m})$, then the Poisson ideal to which it corresponds is an ideal of the form $J_1 + J_2$ where $J_1 = \langle x_{i_1}, \dots, x_{i_r} \rangle$ and J_2 is generated by the images of $x_\alpha - \lambda_\alpha$ in S where $\lambda_\alpha \in \mathbb{C}^\times$ and $\alpha \in \Gamma_{d\tau}$ is such that $\alpha = (\alpha_1, \dots, \alpha_n)$ where $\alpha_{i_1} = \cdots = \alpha_{i_r} = 0$. If $\mathcal{V}(P)$ contains a proper $\sigma_{\mathfrak{m}}$-invariant subvariety $\mathcal{V}$, then J_2 is contained in an ideal J of S generated by certain homogeneous $\sigma_{\mathfrak{m}}$-invariant factors of the generators of J_2. However, such factors multiplied by an appropriate x_α in $\mathbb{C}\Gamma$ have degree zero and are $\sigma_{\mathfrak{m}}$-invariant with eigenvalue one. By Proposition 3.5(c), such an element belongs to Γ_τ, which equals $\Gamma_{d\tau}$ since $\sigma_{\mathfrak{m}}$ has a representative in $\mathrm{GL}(\mathbb{C}^n)$ which belongs to G. It follows from Lemma 1.2 that

J is a Poisson ideal of S so that $\mathcal{V}$ is a point of $\mathcal{V}(P)$. Since $\mathcal{V}(P)$ is connected, the claim is proved.

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