The One-Dimensional Line Schemes of Two Families of Potentially-Generic Quadratic Quantum $\mathbb{P}^3$s

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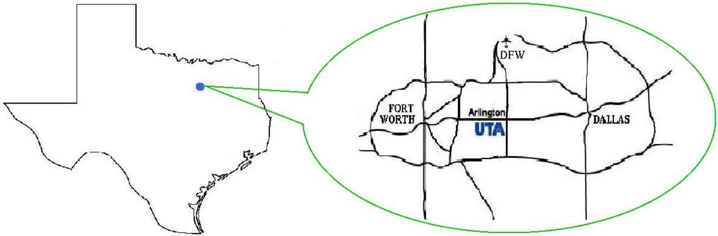

Partial support from NSF DMS-0900239 & DMS-1302050.

Motivation

Quadratic AS-regular Algebras of gldim 4 (field alg closed)

- Van den Bergh (early 1990s): any quad alg on 4 gens with 6 **generic** def rels has 20 nonisom truncated point mods of length 3 (20 is cntd with mult);
  if also AS-reg, then has a 1-param family of line mods. $\langle$ line scheme $\rangle$
- Shelton & Vancliff (late 1990s): any quad alg on 4 gens with 6 def relns that

has a **finite** scheme $\mathfrak{z}$ of trunc point mods of length 3 $\Longrightarrow$ **can recover the def relns from** $\mathfrak{z}$;

or

is AS-reg (+ a few more hyps) & has a **1-diml** line scheme $\mathfrak{L}$ $\Longrightarrow$ **can recover the def relns from** $\mathfrak{L}$.

Goals & Subgoals

Longterm goal: classify all quad AS-reg algs A of gldim 4 using $\mathfrak{z}$ or $\mathfrak{L}$.

Subgoal: identify those $\mathfrak{L}$ of dim $= 1$ where $|\mathfrak{z}| = 20$ (or $|\mathfrak{z}| < \infty$).

My Recent Work

- With R. Chandler: compute & compare $\mathfrak{z}$ & $\mathfrak{L}$ for a certain family of AS-reg algs of gldim 4 that have $|\mathfrak{z}| = 20$ & $\dim(\mathfrak{L}) = 1$.
- With D. Tomlin: same problem for a different family of AS-reg algs of gldim 4 that have $|\mathfrak{z}| = 20$ & $\dim(\mathfrak{L}) = 1$.

Both families of algebras found by Cassidy & Vancliff several years ago & viewed as potentially-generic quadratic quantum $\mathbb{P}^3$s.

Work with Chandler

field $\Bbbk = \bar{\Bbbk}$, $\text{char}(\Bbbk) \neq 2$, i, $\gamma \in \Bbbk^\times$, $i^2 = -1$, gens $= x_1, \ldots, x_4$,

relations:
$$x_4x_1 = ix_1x_4, \quad x_3^2 = x_1^2, \quad x_3x_1 = x_1x_3 - x_2^2,$$
$$x_3x_2 = ix_2x_3, \quad x_4^2 = x_2^2, \quad x_4x_2 = x_2x_4 - \gamma x_1^2.$$

If $\gamma(\gamma^2 - 4) \neq 0$, then $|\mathfrak{z}| = 20$ and can be grouped naturally in $\mathbb{P}^3$ into 2 sets of 2 points and 4 sets of 4 points (call latter type generic points).

If $\text{char}(\Bbbk) = 0$, $\gamma(\gamma^2 - 16) \neq 0$, then $\mathfrak{L} =$ union of 6 subschemes in $\mathbb{P}^5$:

- 1 nonplanar elliptic curve in a $\mathbb{P}^3$ (i.e., spatial elliptic curve),
- 4 planar elliptic curves,
- a subscheme in a $\mathbb{P}^3$ consisting of the union of 2 nonsingular conics.

In $\mathbb{P}^3$, through each of the 16 generic points of $\mathfrak{z}$ passes exactly 6 lines of those lines parametrized by $\mathfrak{L}$ (1 line for each of the above 6 subschemes).

Work (in progress) with Tomlin

field $\mathbb{k} = \bar{\mathbb{k}}$, char($\mathbb{k}$) $\neq 2$, $\alpha \in \mathbb{k}^\times$, gens $= x_1, \ldots, x_4$,

relations:
$$\begin{aligned} x_1x_3 &= -x_3x_1, & x_2x_3 &= x_3x_2, & x_2x_4 + x_4x_2 &= x_3^2, \\ x_1x_4 &= -x_4x_1, & x_2^2 &= x_4^2, & 2x_2^2 + \alpha x_3^2 &= x_1^2. \end{aligned}$$

If $\alpha(\alpha^2 - 1) \neq 0$, then $|\mathfrak{z}| = 20$ and can be grouped naturally in $\mathbb{P}^3$ into 10 sets of 2 points.

If char($\mathbb{k}$) $= 0$, $\alpha(\alpha^2 - 1) \neq 0$, then $\mathfrak{L} =$ union of 6 subschemes in $\mathbb{P}^5$:

- 1 nonplanar elliptic curve in a $\mathbb{P}^3$, (i.e., spatial elliptic curve),
- 1 nonplanar rational curve (with 1 singular point) in a $\mathbb{P}^3$,
- 2 planar elliptic curves,
- 2 subschemes, each of which consists of the union of a nonsingular conic and a line (that meets the conic in 2 distinct points).

To do: find how many lines through each point. Does this distinguish any points?

Conjecture re Line Scheme of Generic Quantum $\mathbb{P}^3$

Conjecture

One of the generic classes of quadratic AS-regular algebra of gldim 4 should have line scheme that consists of the union of 2 spatial elliptic curves and 4 planar elliptic curves.

Some References & Further Reading

- M. Artin, J. Tate & M. Van den Bergh, Some Algebras Associated to Automorphisms of Elliptic Curves, *The Grothendieck Festschrift* **1**, 33-85, Eds. P. Cartier et al., Birkhäuser (Boston, 1990).
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