General conditions for the existence of non-standard Lagrangians for dissipative dynamical systems

Z.E. Musielak

Department of Physics, The University of Texas at Arlington, Arlington, TX 76019, USA[*]

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Equations of motion describing dissipative dynamical systems with coefficients varying either in time or in space are considered. To identify the equations that admit a Lagrangian description, two classes of non-standard Lagrangians are introduced and general conditions required for the existence of these Lagrangians are determined. The conditions are used to obtain some non-standard Lagrangians and derive equations of motion resulting from these Lagrangians.

I. INTRODUCTION

The most concise description of a dynamical system is obtained when its equation of motion can be derived from a Lagrangian by means of the action principle [1-3]. In general, this Lagrangian can be either standard, if it is expressed as the difference between kinetic and potential energy terms [4,5], or non-standard, if its form is such that no clear identification of the kinetic and potential energy terms can be made [3,4,6]; note that non-standard Lagrangians are called non-natural Lagrangians by Arnold [7]. There is also a small group of dynamical systems admitting the so-called alternative Lagrangians [8,9].

Many attempts were made to obtain standard Lagrangians for conservative and non-conservative dynamical systems [3,6, 10-20]. The problem of the existence of standard Lagrangians for non-conservative dynamical systems has a long and interesting history, which began with the original work of Bauer [10] and Bateman [11]. Important contributions were made by many authors, including Dekker [12], Sarlet [13], Riewe [14], Rabei et al. [16], Dreisigmeyer and Young [17], Carinena et al. [19], and many others. In more recent work [20], two general classes of equations of motion that admit standard Lagrangians were identified.

Another interesting problem is identification of equations of motion that admit a Lagrangian description with non-standard Lagrangians. Specific forms of non-standard Lagrangians were proposed for the nonlinear second-order Riccati equation [21] and the Lienard-type nonlinear equation [22,23]. A few special classes of equations of motion that describe dissipative dynamical systems with non-constant coefficients and can be derived from non-standard Lagrangians were also identified [20].

The main purpose of this paper is to generalize the above results by deriving conditions for the existence of non-standard Lagrangians for dissipative dynamical systems with coefficients varying either in time or in space. The presented approach is general and allows obtaining different forms of non-standard Lagrangians and equations of motion resulting from these Lagrangians. Specific examples are presented and discussed.

Let us consider equations of motion of the following form

$$\ddot{x} + P(x,t)\dot{x}^2 + Q(x,t)\dot{x} + R(x,t)x \ = \ \mathcal{F}(t) \ , \tag{1}$$

and the non-standard Lagrangians

$$L(\dot{x},x,t) \ = \ \frac{1}{p(x,t)\dot{x} + q(x,t)x + r(x,t)} \ , \tag{2}$$

where x is a displacement, $\dot{x}$ and $\ddot{x}$ are the first and second derivatives with respect to time t, and the coefficients $P(x,t)$, $Q(x,t)$ and $R(x,t)$, and the driving force $\mathcal{F}(t)$ are arbitrary but continuous, differentiable and integrable functions. In addition, the functions $p(x,t)$, $q(x,t)$ and $r(x,t)$ are to be determined and they must also be continuous and at least twice-differentiable.

General conditions required for the existence of these non-standard Lagrangians are obtained and some specific forms of the equations of motion that admit a Lagrangian description are identified. The outline of the paper is as follows: equations of motion with time and space dependent coefficients that can be derived from the non-standard Lagrangians are determined in Sec. II and III, respectively; and conclusions are given in Sec. IV.

[*]Electronic address: zmusielak@uta.edu

II. DYNAMICAL SYSTEMS WITH TIME-DEPENDENT COEFFICIENTS

A. Non-standard Lagrangians with $h(t) = 0$

Let us consider linear and dissipative dynamical systems described by Eq. (1) with $P(x,t) = 0$, $Q(x,t) = B(t)$, $R(x,t) = C(t)$ and $\mathcal{F}(t) = 0$. The resulting equation of motion describes different oscillators [24], including a damped pendulum whose length increases in time [25]. To determine a non-standard Lagrangian that can be used to derive this equation, we take $p(x,t) = f(t)$, $q(x,t) = g(t)$ and $r(x,t) = h(t)$, and in addition assume that $h(t) = 0$. Conditions that must be satisfied by the functions $f(t)$ and $g(t)$ are given by the following proposition.

Proposition 1: The equation of motion

$$\ddot{x} + B(t)\dot{x} + C(t)x \ = \ 0 \ , \tag{3}$$

can be derived from the non-standard Lagrangian

$$L(\dot{x}, x, t) \ = \ \frac{1}{f(t)\dot{x} + g(t)x} \ , \tag{4}$$

where $B(t)$, $C(t)$, $f(t)$ and $g(t)$ are continuous, differentiable and integrable functions, if, and only if,

$$f(t) \ = \ e^{I_u(t)} \ , \tag{5}$$

$$g(t) \ = \ \frac{2}{3}\left[B(t) - \frac{1}{2}u(t)\right] \ e^{I_u(t)} \ , \tag{6}$$

where

$$I_u(t) \ = \ \int^t u(\tilde{t}) \ d\tilde{t} \ , \tag{7}$$

and $u(t)$ is a solution of the following Riccati equation

$$\dot{u} + \frac{1}{3}u^2 - \frac{1}{3}uB(t) - \left[\frac{2}{3}B^2(t) + 2\dot{B}(t) - 3C(t)\right] = 0 \ . \tag{8}$$

Proof: Using the Euler-Lagrange equation [1-7] and the non-standard Lagrangian $L(\dot{x}, x, t)$ given by Eq. (4), the following equation of motion is obtained

$$\ddot{x} + \frac{1}{2}\left(\frac{\dot{f}}{f} + \frac{3g}{f}\right)\dot{x} + \left(\frac{\dot{g}}{f} - \frac{\dot{f}g}{2f^2} + \frac{g^2}{2f^2}\right)x = 0 \ . \tag{9}$$

Comparison of this equation to Eq. (3) gives two conditions that allow us to determine the functions $f(t)$ and $g(t)$. The conditions can be written as

$$\frac{1}{2}\frac{\dot{f}}{f} + \frac{3}{2}\frac{g}{f} = B(t) \ , \tag{10}$$

and

$$\frac{\dot{g}}{f} - \frac{1}{2}\frac{\dot{f}g}{f^2} + \frac{1}{2}\frac{g^2}{f^2} = C(t) \ . \tag{11}$$

Combining Eqs (10) and (11), we obtain

$$\frac{\ddot{f}}{f} - \frac{2}{3}\left(\frac{\dot{f}}{f}\right)^2 - \frac{\dot{f}}{3f} - \left[\frac{2}{3}B^2(t) + 2\dot{B}(t) - 3C(t)\right] = 0 \ . \tag{12}$$

With $u = \dot{f}/f$, we have

$$\dot{u} + \frac{1}{3}u^2 - \frac{1}{3}uB(t) - \left[\frac{2}{3}B^2(t) + 2\dot{B}(t) - 3C(t)\right] = 0 \ , \tag{13}$$

which is a special form of the Riccati equation (see Eq. 8).

Since

$$f(t) \; = \; e^{I_u(t)} \; , \tag{14}$$

with

$$I_u(t) \; = \; \int^t u(\tilde{t}) \; d\tilde{t} \; , \tag{15}$$

and

$$g(t) \; = \; \frac{2}{3} \left[B(t) - \frac{1}{2} u(t) \right] \; e^{I_u(t)} \; , \tag{16}$$

finding $u(t)$ that satisfies Eq. (13) allows us to determine the functions $f(t)$ and $g(t)$.
To conclude the proof, we comment on the integral in Eq. (15), which can formally be treated as an indefinite integral because any value of the lower limit of this integral can be chosen. Since different indefinite integrals of the same function differ only by an additive constant, the lower limit of the integral can be chosen in such a way that the resulting constant is always zero [26]. Because of this property, the integral is written without its lower limit. Note also that for $u(t)$ being a continuous function, the derivative of the integral with respect to t is given by its integrand [27].

The results of Proposition 1 show that the problem to determine non-standard Lagrangians and derive the equations of motion resulting from these Lagrangians is reduced to finding solutions to the Riccati equation given by Eq. (13). The method is especially useful when applied to equations of motion with the coefficients $B(t)$ and $C(t)$ explicitly given. After substituting these coefficients into the Riccati equation and finding solutions for $u(t)$, it is easy to evaluate the functions $f(t)$ and $g(t)$, and obtain a non-standard Lagrangian. However, it must be noted that in general solutions of the Riccati equation cannot be found by quadratures [28]. Another method to determine non-standard Lagrangians and the corresponding equations of motion is presented in Sec. II.C.

B. Non-standard Lagrangians with $h(t) \neq 0$

We now consider the same equation of motion as that given by Eq. (3), however, assume that the general form of the non-standard Lagrangian $L(\dot{x}, x, t)$ is

$$L(\dot{x}, x, t) \; = \; \frac{1}{f(t)\dot{x} + g(t)x + h(t)} \; , \tag{17}$$

with $h(t) \neq 0$. Substituting this Lagrangian into the Euler-Lagrange equation and comparing the resulting equation of motion to Eq. (3), we obtain

$$\frac{\dot{h}}{f} - \frac{1}{2} \frac{\dot{f} h}{f^2} + \frac{1}{2} \frac{g h}{f^2} = 0 \; , \tag{18}$$

which is a condition to determine $h(t)$. This condition supplements the two other conditions given by Eqs (10) and (11). Using these conditions and Eq. (14), we write Eq. (18) as

$$\dot{h} + \frac{1}{3} \left[B(t) - 2u(t) \right] h \; = 0 \; . \tag{19}$$

It is easy to show that this equation has the following solution

$$h(t) \; = \; e^{2I_u(t)/3} e^{-I_B(t)/3} \; , \tag{20}$$

where

$$I_B(t) \; = \; \int^t B(\tilde{t}) \; d\tilde{t} \; . \tag{21}$$

Since $h(t) = 0$ also satisfies Eq. (18), this implies that there are always two non-standard Lagrangians, one with $h(t) = 0$ and another with $h(t)$ given by Eq. (20), which can be used to derive the same equation of motion.

To demonstrate this, it is enough to substitute both Lagrangians into the Euler-Lagrange equation (see the results presented in Sec. II.C).

So far, we have considered the equation of motion with $\mathcal{F}(t) = 0$ (see Eq. 3). By adding a driving force $\mathcal{F}(t) \neq 0$, the condition given by Eq. (19) becomes

$$\dot{h} + \frac{1}{3}\left[B(t) - \frac{1}{2}u(t)\right] h \ = -\mathcal{F}(t) e^{I_u(t)} \ , \tag{22}$$

and $h(t)$ satisfying the above equation is

$$h(t) = \left[1 - \int^t \mathcal{F}(\tilde{t}) e^{I_u(\tilde{t})/3} e^{I_B(\tilde{t})/3} \ d\tilde{t}\right] e^{2I_u(t)/3} e^{-I_B(t)/3} \ . \tag{23}$$

The main result is that the function $h(t)$ in the non-standard Lagrangian $L(\dot{x}, x, t)$ must be non-zero when $\mathcal{F}(t) \neq 0$. Note also that Eq. (23) reduces to Eq. (20) if it is assumed that $\mathcal{F}(t) = 0$.

C. Special forms of equations of motion

In this paper, we are dealing with general forms of equations of motion for which neither the coefficients $B(t)$ and $C(t)$ nor the driving force $\mathcal{F}(t)$ are explicitly given. As a result, the method of deriving non-standard Lagrangians described in Sec. II.A cannot be used here. However, there is another approach that allows obtaining non-standard Lagrangians and the resulting special equations of motion.

The approach is to specify $f(t)$ and use it to evaluate the functions $u(t)$ and $g(t)$. Substituting $u(t)$ into the Riccati equation, a relationship between $B(t)$ and $C(t)$ is determined; note that this relationship guarantees that the obtained $u(t)$ is a solution of the Riccati equation. As the next step, we determine $h(t)$ and then derive the non-standard Lagrangian and the equation of motion resulting from this Lagrangian.

In the following, we use the approach to derive several special forms of the equations of motion with $\mathcal{F}(t) = 0$. We give the explicit forms of $f(t)$, $g(t)$, $h(t)$ and $u(t)$, and express $C(t)$ in terms of $B(t)$. Here are some interesting cases:
(i) $f(t) = \text{const}$, $u(t) = 0$, $g(t) = \text{const}$, $h(t) = e^{-I_B(t)/3}$ or $h(t) = 0$, and $C = 2B^2/9 = \text{const}$;
(ii) $f(t) = e^{2I_B(t)}$, $u(t) = 2B(t)$, $g(t) = 0$, $h(t) = e^{I_B(t)}$ or $h(t) = 0$ and $C(t) = 0$;
(iii) $f(t) = \alpha = \text{const}$, $u(t) = 0$, $g(t) = 2\alpha B(t)/3$, $h(t) = e^{-I_B(t)/3}$ or $h(t) = 0$, and

$$C(t) = \frac{2}{3}\left[\dot{B}(t) + \frac{1}{3}B^2(t)\right] \ , \tag{24}$$

note that for $B(t) = \text{const}$, the results are reduced to those given by (i);
(iv) $f(t) = e^{I_B(t)/2}$, $u(t) = B(t)/2$, $g(t) = B(t)e^{I_B(t)/2}/2$, $h(t) = \text{const}$ or $h(t) = 0$, and

$$C(t) = \frac{1}{2}\left[\dot{B}(t) + \frac{1}{2}B^2(t)\right] \ ; \tag{25}$$

(v) $f(t) = e^{-I_B(t)}$, $u(t) = -B(t)$, $g(t) = B(t)e^{-I_B(t)}$, $h(t) = e^{-I_B(t)}$ or $h(t) = 0$, and

$$C(t) = \dot{B}(t) \ ; \tag{26}$$

(vi) $f(t) = g(t) = e^{2I_B(t)} e^{-3t}$, $u(t) = 2B(t) - 3$, $h(t) = e^{I_B(t)} \ e^{-2t}$ or $h(t) = 0$, and

$$C(t) = B(t) - 1 \ ; \tag{27}$$

(vii) $f(t) = -g(t) = e^{2I_B(t)} e^{3t}$, $u(t) = 2B(t) - 3$, $h(t) = e^{I_B(t)} \ e^{2t}$ or $h(t) = 0$, and

$$C(t) = -B(t) - 1 \ ; \tag{28}$$

(viii) $f(t) = e^{\int^t d\tilde{t}/v(\tilde{t})}$, with

$$v(t) = e^{-I_B(t)/3} + e^{-I_B(t)/3} \int^t e^{I_B(\tilde{t})/3} d\tilde{t} \ , \tag{29}$$

gives $u(t) = 1/v(t)$, $g(t) = [2B(t) - 1/v(t)]f(t)/3$, $h(t) = e^{-I_B(t)/3}\ f^{2/3}(t)$ or $h(t) = 0$, and the same $C(t)$ is obtained as that given by Eq. (24), which is an interesting result;
(ix) $f(t) = e^{2I_B(t)} - 3\beta e^{2I_B(t)} \int^t e^{-2I_B(\tilde{t})} d\tilde{t}$, $u(t) = 2B(t) + 3\beta/f(t)$, $g(t) = \beta$, and either

$$h(t) = e^{I_B(t)}\ e^{-2\beta \int^t \frac{d\tilde{t}}{f(\tilde{t})}} \ , \tag{30}$$

or $h(t) = 0$, and

$$C(t) = \frac{\beta}{f(t)} \left[\frac{2\beta}{f(t)} - B(t) \right] \ , \tag{31}$$

where β is a constant.

All the above examples are valid for $\mathcal{F}(t) = 0$. They can easily be extended to the equations of motion with $\mathcal{F}(t) \neq 0$ by using the same $f(t)$, $u(t)$ and $g(t)$, expressing $C(t)$ in terms of $B(t)$, and calculating $h(t)$ from Eq. (9). Obviously, the approach can be used to obtain more examples of special equations of motion.

The main difference between the earlier work on non-standard Lagrangians described in [23] and the results presented here is that the previous work was restricted to a special case of $f(t) = 1$ (see the above case (iii) with $\alpha = 1$), whereas the results of this paper are valid for a general case of non-constant $f(t)$.

III. DYNAMICAL SYSTEMS WITH SPACE-DEPENDENT COEFFICIENTS

A. Non-standard Lagrangians with $H(x) = 0$

We now consider a class of dissipative dynamical systems described by Eq. (1) with $P(x,t) = a(x)$, $Q(x,t) = b(x)$, $R(x,t) = c(x)$ and $\mathcal{F}(t) = 0$. The obtained equation of motion describes different oscillators [24] including a Liénard oscillator [20,21]. A non-standard Lagrangian that can be used to derive this equation is obtained by assuming that $p(x,t) = F(x)$, $q(x,t) = G(x)$ and $r(x,t) = H(x)$. We begin with $H(x) = 0$ and then consider a more general case of $H(x) \neq 0$. The conditions that must be satisfied by the functions $F(x)$ and $G(x)$ are given by the following proposition.

Proposition 2: The equation of motion

$$\ddot{x} + a(x)\dot{x}^2 + b(x)\dot{x} + c(x)x \ = \ 0 \ , \tag{32}$$

can be derived from the non-standard Lagrangian

$$L(\dot{x}, x) \ = \ \frac{1}{F(x)\dot{x} + G(x)x} \ , \tag{33}$$

where $a(x)$, $b(x)$, $c(x)$, $F(x)$ and $G(x)$ are continuous, differentiable and integrable functions, if, and only if,

$$F(x) \ = \ e^{I_a(x)} \ , \tag{34}$$

$$G(x) \ = \ 3\frac{c(x)}{b(x)} e^{I_a(x)} \ , \tag{35}$$

with

$$I_a(x) \ = \ \int^x a(\tilde{x}) d\tilde{x} \ , \tag{36}$$

and

$$\frac{2}{3} b(x) \ = \ \frac{d}{dx} \left[3\frac{c(x)}{b(x)} x \right] + \left[3\frac{c(x)}{b(x)} x \right] a(x) \ . \tag{37}$$

Proof: We substitute the non-standard Lagrangian $L(\dot{x}, x)$ given by Eq. (33) into the Euler-Lagrange equation [1-7] and obtain

$$\ddot{x} + \frac{F'(x)}{F(x)} \dot{x}^2 + \frac{3}{2F(x)} \left[G'(x)x + G(x) \right] \dot{x}$$

$$+\frac{1}{2F^2(x)}\left[G'(x)x+G(x)\right]G(x)\ x\ =0\ , \quad (38)$$

where $F'(x)=dF(x)/dx$ and $G'(x)=dG(x)/dx$. Comparison of the above equation to Eq. (32) gives

$$\frac{F'(x)}{F(x)}=a(x)\ , \quad (39)$$

$$\frac{3}{2F(x)}\left[G'(x)x+G(x)\right]=b(x)\ , \quad (40)$$

and

$$\frac{1}{2F^2(x)}\left[G'(x)x+G(x)\right]G(x)=c(x)\ , \quad (41)$$

which can be written as

$$F(x)\ =\ e^{I_a(x)}\ , \quad (42)$$

and

$$G(x)\ =\ 3\frac{c(x)}{b(x)}e^{I_a(x)}\ , \quad (43)$$

with

$$I_a(x)\ =\ \int^x a(\tilde{x})d\tilde{x}\ . \quad (44)$$

Having expressed the functions $F(x)$ and $G(x)$ in terms of the coefficients $a(x)$, $b(x)$ and $c(x)$, we now determine a relationship that must be satisfied by these coefficients in order to admit a Lagrangian description. Using Eqs (42) and (43), we write

$$L(\dot{x},x)\ =\ \frac{e^{-I_a(x)}}{\dot{x}+3\frac{c(x)}{b(x)}x}\ . \quad (45)$$

Substituting this Lagrangian into the Euler-Lagrange equation, we obtain

$$\ddot{x}+a(x)\dot{x}^2+\frac{3}{2}\left[\frac{d}{dx}\left(3\frac{c(x)}{b(x)}x\right)+\left(3\frac{c(x)}{b(x)}x\right)a(x)\right]\dot{x}$$

$$+\frac{3}{2}\frac{c(x)}{b(x)}\left[\frac{d}{dx}\left(3\frac{c(x)}{b(x)}x\right)+\left(3\frac{c(x)}{b(x)}x\right)a(x)\right]x\ =\ 0\ . \quad (46)$$

In order for this equation to be the same as Eq. (32), the following condition must be satisfied

$$b(x)\ =\ \frac{3}{2}\left[\frac{d}{dx}\left(3\frac{c(x)}{b(x)}x\right)+\left(3\frac{c(x)}{b(x)}x\right)a(x)\right]\ . \quad (47)$$

Derivation of this condition concludes the proof of Proposition 2.

An important result of Proposition 2 is that only those equations of motion whose coefficients satisfy the above condition can be derived from the non-standard Lagrangian given by Eq. (45). Clearly, the equations of motion that admit Lagragian description must be of very special forms (see Sec. III.C). Note also that the condition given by Eq. (47) reduces to that obtained and discussed in [23] when it is assumed that $a(x)=0$.

B. Non-standard Lagrangians with $H(x) \neq 0$

We consider the same equation of motion as that given by Eq. (32) and the following non-standard Lagrangian

$$L(\dot{x}, x) = \frac{1}{F(x)\dot{x} + G(x)x + H(x)} , \tag{48}$$

in which $H(x) \neq 0$. By substituting this Lagrangian into the Euler-Lagrange equation, we derive Eqs. (34) and (36), and in addition obtain

$$G'(x)x + G(x) + H'(x) = \frac{2}{3}F(x)b(x) , \tag{49}$$

$$[G'(x)x + G(x) + H'(x)]G(x) + G'(x)H(x) = 2F^2(x)c(x) , \tag{50}$$

and

$$[G(x) + H'(x)]H(x) = 0 . \tag{51}$$

Since $H(x) \neq 0$, Eq. (51) gives $G(x) = -H'(x)$ and we have

$$G'(x) = \frac{2}{3}\frac{b(x)}{x}e^{I_a(x)} , \tag{52}$$

and

$$G(x) = \frac{2}{3}\int^x \frac{b(\tilde{x})}{\tilde{x}}e^{I_a(\tilde{x})}d\tilde{x} . \tag{53}$$

Using the above equations, we find

$$H(x) = 3\frac{c(x)}{b(x)}xe^{I_a(x)} - G(x)x , \tag{54}$$

which gives

$$L(\dot{x}, x) = \frac{1}{F(x)\dot{x} + G(x)x} . \tag{55}$$

This is an interesting result as it shows that the function $H(x)$ does not play any role in $L(\dot{x}, x)$ because the form of the above Lagrangian is the same as that given by Eq. (33). The consequence is that the results of Proposition 2 are also valid for $H(x) \neq 0$.

C. Special equations of motion

We now present several special equations of motion. Let us begin with the assumption that $b(x) = x$ and $c(x) = x$. The results of Proposition 2 demonstrate that the non-standard Lagrangian (see Eq. 33) exists only when $a(x) = 2/9 - 1/x$. This restriction on the form of the coefficients $a(x)$, $b(x)$ and $c(x)$ results from the relationship given by Eq. (37). In case the coeffcients do not satisfy the relationship, the non-standard Lagrangian $L(\dot{x}, x)$ does not exist. Note also that if $a(x) = 0$ and $b(x) = x$, then $L(\dot{x}, x)$ exists only if $c(x) = x^2/9$. This shows how special equations of motion must be in order to admit the Lagrangian description with the non-standard Lagrangians.

Other special forms of equations of motion can be derived by specifying the function $G(x)$ and expressing the coefficient $c(x)$ in terms of $b(x)$ and $a(x)$. Then, Eq. (37) is used to determine $b(x)$ as a function of $a(x)$. In this approach, $a(x)$ is arbitrary as long as it is a continuous, differentiable and integrable function. Here are several specific examples:

(i) $G(x) = 1$ leads to the following non-standard Lagrangian

$$L(\dot{x}, x) = \frac{1}{\dot{x}e^{I_a(x)} + x} , \tag{56}$$

and the equation of motion with $c(x) = b(x)e^{-I_a(x)}/3$, where $b(x) = 3e^{-I_a(x)}/2$;
(ii) $G(x) = 1/x$ gives the following non-standard Lagrangian

$$L(\dot{x}, x) = \frac{1}{\dot{x}e^{I_a(x)} + 1} , \tag{57}$$

and the equation of motion with $b(x) = 0$ and $c(x) = 0$ (see also [23]);
(iii) $G(x) = x$ leads to the following non-standard Lagrangian

$$L(\dot{x}, x) = \frac{1}{\dot{x}e^{I_a(x)} + x^2} , \tag{58}$$

and the equation of motion with $c(x) = b(x)xe^{-I_a(x)}/3$, where $b(x) = 3xe^{-I_a(x)}$;
(iv) $G(x) = 3e^{I_a(x)}$ gives the following non-standard Lagrangian

$$L(\dot{x}, x) = \frac{e^{-I_a(x)}}{\dot{x} + 3x} , \tag{59}$$

and the equation of motion with $c(x) = b(x) = 9[1 + xa(x)]/2$.

An interesting result is that in the case (iii) with $a(x) = 0$, one obtains $b(x) = 3x$, $c(x) = x^2$ and the resulting non-standard Lagrangian

$$L(\dot{x}, x) = \frac{1}{\dot{x} + x^2} , \tag{60}$$

gives $\ddot{x} + 3x\dot{x} + x^3 = 0$, which is Duffing's equation [6,29] with a non-constant damping coefficient.

IV. DISCUSSION AND CONCLUSIONS

The main purpose of this paper was to derive general conditions for the existence of non-standard Lagrangians for dissipative dynamical systems. Two separate classes of these systems were considered, namely, the systems whose coefficients depend only on time and those with coefficients depedning solely on space.

The systems with time-dependent coefficients are described by the equation of motion: $\ddot{x} + B(t)\dot{x} + C(t)x = 0$. A general form of non-standard Lagrangians that can be used to derive this equation is $L(\dot{x}, x, t) = 1/[f(t)\dot{x} + g(t)x + h(t)]$. An important result of this paper is that $L(\dot{x}, x, t)$ exists only if $f(t)$, or more specifically $u(t) = \dot{f}(t)/f(t)$, satisfies the Riccati equation given by Eq. (8), and that the same equation of motion is obtained regardless whether $h(t) = 0$ (see Proposition 1) or $h(t) \neq 0$ (see Eq. 20). This means that there are always two different non-standard Lagrangians that give the same equation of motion.

The situation is different when the considered dynamical system is driven and the equation of motion describing this system is given by $\ddot{x} + B(t)\dot{x} + C(t)x = \mathcal{F}(t)$. The main result is that the function $h(t)$ in the non-standard Lagrangian $L(\dot{x}, x, t) = 1/[f(t)\dot{x} + g(t)x + h(t)]$ must be non-zero if $\mathcal{F}(t) \neq 0$. The procedure of finding the functions $f(t)$ and $g(t)$ is the same as that described in Proposition 1, however, the function $h(t)$ must be now determined using Eq. (23).

It was also shown that the method of finding $L(\dot{x}, x)$ presented in this paper can be applied to the equations of motion whose coefficients $B(t)$ and $C(t)$ are either explicitly given or arbitrary. Several examples of equations of motion with arbitrary coefficients are presented in Sec. II.C. The obtained results clearly show that only equations of motion whose coefficients $B(t)$ and $C(t)$ are related to each other admit the Lagrangian description.

The systems with space-dependent coefficients considered in this paper are described by the equation of motion: $\ddot{x} + a(x)\dot{x}^2 + b(x)\dot{x} + c(x)x = 0$. The general form of non-standard Lagrangians that can be used to derive this equation is $L(\dot{x}, x) = 1/[F(x)\dot{x} + G(x)x + H(x)]$. Our main result is that after evaluating the functions $F(x)$ and $G(x)$ the Lagrangian becomes $L(\dot{x}, x) = e^{-I_a(x)}/[\dot{x} + 3xc(x)/b(x)]$, regardless whether we take $H(x) = 0$ or $H(x) \neq 0$. This independece of the non-standard Lagrangian on $H(x)$ makes the results of Proposition 2 also valid for $H(x) \neq 0$. An important result of Proposition 2 is that only those equations of motion whose coefficients satisfy the condition given by Eq. (47) can be derived from the above non-standard Lagrangian. Several non-standard Lagrangians and the resulting equations of motion presented in Sec. III.C show how special forms these equations must have in order to admit the Lagrangian description.

The results presented in this paper generalize the previous work [23] in which $f(t) = 1$, $F(x) = 1$ and $a(x) = 0$ were considered, except for one special case of $F(x) = exp(I_a(x))$, with $a(x)$ being arbitrary, and $G(x) = 1/x$ (see the case (ii) in Sec. II.C). Our generalization is to allow $f(t)$ and $F(x)$ to be arbitrary and obtaining general conditions that can be used to determine these functions and, in addition, deriving the relationship that relates the coefficients $a(x)$, $b(x)$ and $c(x)$.

Finally, it must be mentioned that the existence of non-standard Lagrangians for some dissipative dynamical systems has important physical implications. The main advantage of this Lagrange formulation is that it guarantees a well-formulated dynamical problem and, therefore, it is central to theory of these dynamical systems. Knowing the non-standard Lagrangians, the action can be determined and the action principle [2,4] can formally be used to develop the theory. Significance of this action lies in the fact that it becomes a generating functional for the physical properties of the considered dynamical systems [3].

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